PERIODIC QUASIFLATS IN HIERARCHICALLY HYPERBOLIC SPACES

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Abstract. We prove a quasiflat closing theorem and a coarse flat torus theorem for hierarchically hyperbolic groups (HHGs). Namely, given an HHG G, we prove that G is hyperbolic if and only if it contains no $\mathbb{Z}^2$ subgroups and, if $A \leqslant G$ is virtually $\mathbb{Z}^n$, then there is an A–invariant n–dimensional uniform quality quasiflat F such that any two points in F are joined by a uniform-quality hierarchy path lying in F. The later is a consequence of a more detailed theorem describing a "coarse minset" for A in G, which has various applications, including an ascending chain condition for virtually abelian subgroups, hierarchical quasiconvexity of highest abelian subgroups, and some geometric control over normalisers, centralisers, and commensurators of abelian subgroups. We furthermore use this to rule out HHG structures for certain Coxeter groups on the basis of their affine subgroups, and to give a new proof that virtually solvable subgroups of HHGs are virtually abelian, which simplifies the original proof by avoiding Gromov's polynomial growth theorem.

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1. Introduction

A common feature among spaces with some form of (coarse) non-positive curvature is that virtually abelian subgroups of their isometry groups are often particularly well-behaved. Here are two classical illustrations of this phenomenon. First, assume X is a Gromov-hyperbolic space. If A is a virtually abelian group acting by semisimple isometries on X, then there is either an A–invariant, uniformly bounded subset of X or an A-invariant uniform quality quasiline on which A acts cocompactly. In particular, if the action of A is proper, then A is virtually cyclic with undistorted orbits. Second is the Flat Torus Theorem for CAT(0) spaces, which states (in part): if X is a CAT(0) space and A is a virtually abelian group acting on X by semisimple isometries then there is an A–invariant closed, convex flat $F \subseteq X$ on which A acts cocompactly. At this level of generality, this was proven in [BH99], generalising earlier results about Hadamard manifolds [GW71, LY72].

Among the CAT(0) spaces widely studied in group theory are CAT(0) cube complexes. Chepoi famously characterised CAT(0) cube complexes as those cubical complexes whose 1–skeletons are median graphs, and, in fact, if X is a CAT(0) cube complex, then equipping each cube with the ℓ_1 metric yields a path metric d that is median, and extends the median metric on the 0–skeleton given by Chepoi's theorem [Che00, Mie14]. A subspace Y of X that is closed under taking medians is a *median subalgebra* (the terminology reflects the correspondence between CAT(0) cube complexes and discrete median algebras [Rol98]). If the median subalgebra Y is a connected subcomplex, then Y is itself a CAT(0) cube complex, and is *path-isometric* in the sense that any two points in Y are joined by a geodesic of X lying in Y [HP22, Lem. 2.11].

Haglund showed in [Hag23b] that any $g \in \mathrm{Isom}(X, d)$ either fixes the barycentre of some cube, or has a d–geodesic axis F — i.e. a connected median subalgebra isometric to $\mathbb{R}$

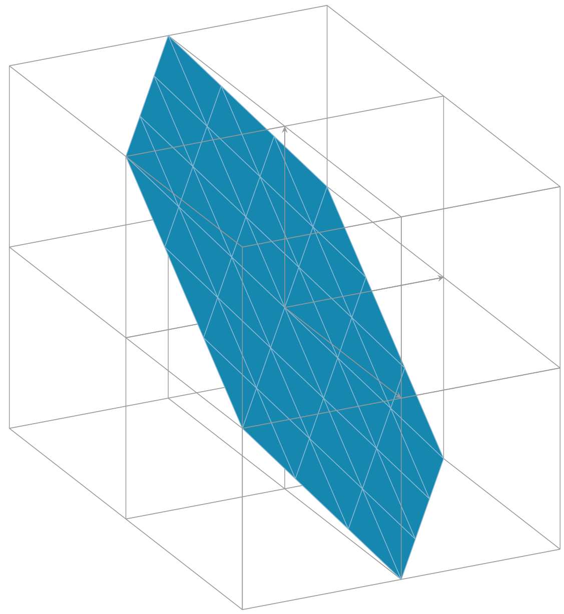

FIGURE 1. The $(3,3,3)$ Coxeter group acts on the standard tiling X of $\mathbb{E}^3$ by unit cubes, preserving a 2–dimensional flat F that crosses every hyperplane. The smallest median subalgebra of X containing F is the whole of X.

— along which it acts as a translation[1]. This can be viewed as a 1–dimensional flat torus theorem for (X,d), since the $\langle g\rangle$–action on F is cocompact. The situation becomes more delicate for higher-rank (virtually) abelian subgroups $A \leqslant \mathrm{Isom}(X,d)$. On one hand, in [Woo17], Woodhouse generalised Haglund's result by showing that if A is virtually $\mathbb{Z}^n$, and the A–action is proper, then (up to subdividing to remove inversions) A stabilises a path-isometric subcomplex $Y \subseteq X$, where Y is cubically isomorphic to $\prod_{i=1}^m C_i$ for $m \geqslant n$ and $C_1,\ldots,C_m$ a collection of CAT(0) cube complexes quasi-isometric to $\mathbb{R}$. Note that Woodhouse's result does not assert that A acts on Y cocompactly, and indeed it may be that no such Y is cocompact. It is implicit in Woodhouse's proof that Y can be taken to be a median subalgebra. In fact, Genevois elaborates on Woodhouse's result in [Gen22], using the language of invariant median subalgebras. However, any theorem providing an A–invariant median subalgebra must differ from the Flat Torus Theorem in one quite serious respect: one cannot in general expect A–cocompactness.

Indeed, Figure 1 illustrates the case where A is the $(3,3,3)$ Coxeter group, and X is the standard tiling of $\mathbb{E}^3$ by unit 3–cubes; the action of A on X comes from applying Sageev's construction to the set of walls in $\mathbb{E}^2$ provided by the A–translates of the three reflection lines corresponding to the generators of A. Applying the CAT(0) flat torus theorem, and using the general fact that CAT(0)-convex subsets of a CAT(0) cube complex are d–isometric (though not in general d–convex) yields an A–cocompact d–isometric subspace $F \subseteq X$ which, equipped with the CAT(0) metric, is isometric to $\mathbb{E}^2$. Moreover, by perturbing F equivariantly, one can make it into an A–cocompact 2–dimensional d–isometric subcomplex Y, homeomorphic to $\mathbb{R}^2$. However, as illustrated in Figure 1, which shows F, no median subalgebra of X is Hausdorff-close to F.

Nonetheless, under reasonable conditions, one can find A–cocompact bilipschitz flats in the median setting by relaxing the median subalgebra requirement, using instead d–isometricity.

[1]One must subdivide to arrange for there to be such an axis in the 1–skeleton.

This is a consequence of the CAT(0) result, and the proof is short enough to present here, to motivate the subsequent discussions.

Proposition 1.1. *Let (Q, d, μ) be a complete, connected, finite-rank median metric space and let $A \leqslant \mathrm{Isom}(Q)$ be a finitely generated virtually abelian subgroup acting properly, and let r be the rank of A. Then there is a proper cocompact action of A on $\mathbb{E}^r$ and an A–equivariant bilipschitz embedding $f : \mathbb{E}^r \to Q$ such that any two points in $f(\mathbb{E}^r)$ are joined by a d–geodesic lying in $f(\mathbb{E}^r)$.*

Proof. By [Bow16, Thm. 1.1, Thm. 8.3], there is a complete CAT(0) metric σ on Q and a bilipschitz homeomorphism $h : (Q, \sigma) \to (Q, d)$ that sends geodesics to reparameterised geodesics (as a map of sets, h is just the identity). Moreover, $\mathrm{Isom}(Q, d)$ acts on (Q, σ) by isometries and h is $\mathrm{Isom}(Q, d)$–equivariant. By, for instance, [Fio24, Cor. A], infinite order elements of A have positive stable translation length on (Q, d), so since h is a bilipschitz map, A acts on (Q, σ) semisimply. Hence the Flat Torus Theorem [BH99, Cor. II.7.2] provides a cocompact A–action on $\mathbb{E}^r$ and an A–equivariant isometric embedding $e : \mathbb{E}^r \to (Q, \sigma)$ with σ–convex image, and we conclude by taking $f = h \circ e$. □

Remark 1.2. We emphasise that Q need not be a locally compact space, and throughout this paper, we rarely make such an assumption. Hence, wherever we talk about *proper* maps or *proper* actions, we are referring to metric properness. ✽

When Q is a CAT(0) cube complex, the map f from Proposition 1.1 can be chosen so that $f(\mathbb{E}^r)$ is a d–isometric subcomplex of Q. We view Proposition 1.1 as yet another reason to study such subcomplexes of CAT(0) cube complexes, which have recently received renewed attention in the very interesting paper [BCDK26] on so-called *ample* subsets of $Q^{(0)}$, which correspond to 0–skeletons of d–isometric subcomplexes.[2] In some sense, the present paper is about a "coarse-ification" of d–isometricity in the setting of hierarchically hyperbolic spaces.

Hierarchically hyperbolic spaces (HHSs) were introduced in [BHS17a, BHS19] to capture the similarities in the large-scale structure of mapping class groups and compact special groups. These spaces are, in essence, highly organised examples of coarse median spaces, in the sense of Bowditch [Bow13]. The class of HHSs includes all hyperbolic spaces and is closed under taking products and under relative hyperbolicity (i.e. if X is hyperbolic relative to HHS subspaces then X is an HHS) [BHS19, Prop. 8.27, Thm. 9.3], but one can weaken the bounded penetration property of relative hyperbolicity to allow for the peripheral subspaces to interact in a coarse product structure, allowing the class to accommodate, for example, all right-angled Artin groups. See [BHS19] for the formal definition in its modern form.

The fact that HHSs are coarse median spaces is witnessed by the (strictly stronger) cubical approximation theorem [BHS21, Thm. 2.1], which states: for any n points $\{x_1, \dots, x_n\}$ in an HHS, there is a coarsely median-preserving quasi-isometric embedding, with constants depending only on the HHS structure and n, from a CAT(0) cube complex to the hull (see Subsection 2.8) of $\{x_1, \dots, x_n\}$, such that each x_i is in the image of the 0-skeleton. In a large subclass of HHSs, this local property becomes global: Petyt showed that any colourable hierarchically hyperbolic group (HHG) admits a quasi-isometry to a CAT(0) cube complex [Pet26].

Geodesics in HHSs are not in general closely related to the HHS structure. Instead, one usually works with hierarchy paths, which are quasigeodesics that do respect the HHS

[2] A particularly attractive feature of the notion is that it can be characterised in non-metric terms, provided Q is a finite-dimensional CAT(0) cube complex: the subcomplex $Y \subseteq Q$ is d–isometric if and only if $Y \cap h_1 \cap \dots \cap h_k$ is empty or connected whenever $h_1, \dots, h_k$ are hyperplanes of Q.

structure (see Definition 2.24). Hierarchical quasiconvexity is defined in terms of these paths: a subspace Y of an HHS X is hierarchically quasiconvex (HQC) if every hierarchy path between points in Y is close to Y (of course, this should be quantified appropriately; see Definition 2.22). Hierarchy paths are analogues of ℓ^1-geodesics in CAT(0) cube complexes, so hierarchical quasiconvexity is the analogue of median convexity in CAT(0) cube complexes. In light of the preceding discussion, a flat torus theorem for HHSs will need to involve a notion of convexity which imitates CAT(0) convexity, or ℓ_1–metric isometricity, instead.

For this purpose, we introduce the following notion: a subspace Y of an HHS X is *existentially hierarchically quasiconvex (EHQC)* if, for all $x, y \in Y$ there is a hierarchy path in a uniform neighbourhood of Y which joins x to y (again, this should be parametrised correctly; see Definition 2.30). As in the results about CAT(0) and hyperbolic spaces, we also need a semisimplicity assumption, provided by Definition 5.1.

A simplified version of our "HHS flat torus theorem" is the following, which answers the special cases of Questions 6.1 and 6.2 in [AHPZ25] concerning HHGs:

Theorem 1.3 (Theorem 6.1.(2))**.** *Let $n \in \mathbb{N}$ and let A be a virtually $\mathbb{Z}^n$ group. Suppose A is a subgroup of an HHG $(X, \mathfrak{S})$ — or, more generally, that A acts properly and hierarchically semisimply by HHS automorphisms on an HHS $(X, \mathfrak{S})$. Then there is a proper cocompact action of A on the Euclidean space $\mathbb{E}^n$, an A–invariant EHQC subspace $F \subseteq X$, and an A–equivariant quasi-isometry $F \to \mathbb{E}^n$. Moreover, the EHQC and quasi-isometry constants depend only on X.*

We prove a more detailed statement in Theorem 6.1. In particular, we describe the coarse structure of the union of a natural family of A-invariant, uniform quality EHQC quasiflats, proving that this subspace is A-equivariantly quasi-isometric to a product $Y \times C \times \mathbb{E}^n$, where Y is an injective space with a trivial A-action and C is a CAT(0) space with a trivial A-action. In the case where X is the Cayley graph of an HHG G and $A \leqslant G$, this subspace can be taken to be the normaliser of A. Without the hierarchical semisimplicity assumption, the theorem fails, as is already evident in the case where X is hyperbolic and A is generated by a single parabolic isometry.

In [FFMS25, Thm. 3.15], Fournier-Facio, Mangioni and Sisto prove that, if G is a hierarchically hyperbolic group (HHG), K is finitely generated and $K \hookrightarrow G \twoheadrightarrow E$ is a central extension, then the corresponding element of $H^2(E, K)$ is represented by a bounded cocycle. We generalise this to the setting of hierarchically semisimple actions on HHSs (in particular, the following theorem applies to all subgroups of HHGs).

Theorem 6.5. *Let $(X, \mathfrak{S})$ be an HHS and let $B \leqslant \mathrm{Isom}(X)$ act by HHS automorphisms. Suppose $A \trianglelefteq B$ is a normal finitely generated infinite abelian subgroup, and that the action of A on $(X, \mathfrak{S})$ is proper and hierarchically semisimple. Then there are finite index subgroups $A' \leqslant A$ and $B' \leqslant B$ such that A' is central in B', and the element of $H^2(B'/A', A')$ corresponding to the central extension $A' \hookrightarrow B' \twoheadrightarrow B'/A'$ is represented by a bounded cocycle.*

Gersten showed [Ger92] that if $K \hookrightarrow G \twoheadrightarrow E$ is a central extension represented by a bounded cocycle, then G is naturally quasi-isometric to the direct product $K \times E$, but the converse to this false in general [FS23], even for CAT(0) groups [AM24, Thm. 1.1].

Remark 1.4 (Virtual direct products)**.** The CAT(0) Flat Torus Theorem includes a stronger version of Theorem 6.5 in the free abelian case: suppose B is a finitely generated group acting by isometries on a CAT(0) space X and suppose $A \trianglelefteq B$ is a normal subgroup whose action on X is proper and by semisimple isometries. Then there is a finite index subgroup $B' \leqslant B$ which contains A as a direct factor [BH99, Thm. II.7.1.(5)].

The corresponding statement for actions on HHSs cannot hold, as we now illustrate using mapping class groups. Let Σ be a connected orientable surface of finite type, and suppose that Σ is either closed with genus $\geqslant 3$, or has genus $\geqslant 2$ and at least one boundary component or two punctures. Then the mapping class group $\mathcal{MCG}(\Sigma)$ contains finitely generated subgroups that split as central extensions but are not virtually direct products (see [BH99, Thm. II.7.26]). Since mapping class groups are HHGs, and in particular act properly and hierarchically semisimply by HHS automorphisms on HHSs, this implies that Theorem 6.5 is optimal in the sense that the conclusion cannot be improved to give virtually split extensions as in the CAT(0) case. ✽

As in the CAT(0) case, our version of the flat torus theorem has many applications. To begin with, HHGs (in fact, a more general class of groups) satisfy the ascending chain condition for virtually abelian subgroups:

Corollary 7.3 (Ascending chain condition)**.** *Let G be a group and let $H_1 \leqslant H_2 \leqslant \cdots \leqslant G$ be an ascending chain of finitely generated virtually abelian subgroups. If G acts uniformly properly and hierarchically semisimply by isometric HHS automorphisms on an HHS $(X, \mathfrak{S})$, then $H_n = H_{n+1}$ for all sufficiently large n.*

In particular, any virtually abelian subgroup $H \leqslant G$ is finitely generated and virtually contained in a highest virtually abelian subgroup.

Given $n \in \mathbb{N}$, a virtually $\mathbb{Z}^n$ subgroup $A \leqslant G$ of a group G is *highest* if no finite index subgroup of A is contained in a subgroup of G isomorphic to $\mathbb{Z}^{n+1}$.

In [DHS17] it is stated that, if G is a hierarchically hyperbolic group, then any $H \leqslant G$ either contains F_2 or is virtually abelian; the proof given in [DHS17] had a gap which is corrected in [DHS20] assuming finite generation of H; Corollary 7.3 allows one to remove the finite generation hypothesis (apply the result from [DHS20] to finitely generated subgroups $H_n \leqslant H \leqslant G$ with $\bigcup_n H_n = H$). Hence the version of the Tits alternative stated in [DHS17, Thm. 9.15] holds as stated.

Relatedly, we give a new proof that virtually solvable subgroups of HHGs are virtually abelian which, unlike that in [DHS17, DHS20], avoids Gromov's polynomial growth theorem:

Corollary 7.4. *Let $(X, \mathfrak{S})$ be an HHS and let G be a group acting uniformly properly and hierarchically semisimply by HHS automorphisms on X. If G is virtually solvable then G is virtually abelian.*

Recall that, in the CAT(0) cubical setting, abelian subgroups of cubulated groups may not be convex-cocompact, but Wise–Woodhouse showed [WW17] that, if G acts geometrically on a CAT(0) cube complex X, and $A \leqslant G$ is a highest virtually abelian subgroup, then there is a convex, A–cocompact subcomplex $Y \subseteq X$. In exact analogy, we show:

Corollary 7.8. *Let $(G, \mathfrak{S})$ be an HHG. Every highest virtually abelian subgroup $A \leqslant G$ is hierarchically quasiconvex. Thus every abelian subgroup is virtually contained in a hierarchically quasiconvex abelian subgroup.*

In the important case where G is the mapping class group of a finite-type hyperbolic surface, abelian subgroups were well-understood even before the introduction of curve graphs, subsurface projections, and the "hierarchical" viewpoint. For example, Corollary 7.4, and the upper bound on rank for free abelian subgroups, was established in [BLM83] using Thurston's classification of mapping classes, systems of reducing curves, etc. Although the notions of HQC and EHQC subspaces/subgroups cannot be defined without first introducing some "hierarchical" ideas — enough to be able to talk about coarse medians, for instance — the extra input of Thurston theory seems to simplify proofs like that of, say, Corollary 7.8 in the mapping class group case. However, as pointed out to us by Jason Behrstock, our results recover a useful information about virtually abelian subgroups in the mapping class group case without recourse to the powerful Thurston classification.[3]

Corollary 7.8 relies on a stronger result, Corollary 7.6, showing that the normaliser $N_G(A)$ is EHQC in G and that there is a finite-index subgroup $A_1 \leqslant A$ whose centraliser $C_G(A_1)$ is hierarchically quasiconvex (which, recall, is stronger than EHQC). Corollary 7.8 combines with results in [HHP23, Hod23] to limit the possible point groups of highest virtually abelian subgroups of HHGs, which we make precise in Corollary 7.9. In Corollary 7.11, we apply this to Coxeter groups; specifically, we rule out HHG structures on Coxeter groups containing certain "poison" affine subgroups. We suspect that Coxeter groups without "poison" affine subgroups are HHGs, and that this is closely related to the question of cocompact cubulation. It is not currently known exactly which Coxeter groups are cocompactly cubulated, although progress on a characterisation is provided by [NR03, Wil98, FLS24, HW10]. In view of Remark 7.10, Corollary 7.11, and existing results, we conjecture the following.

Conjecture 1.5. *A Coxeter group is an HHG if and only if it is cocompactly cubulated.*

Corollary 7.6 and Lemma 2.28 combine to yield another interesting consequence:

Corollary 1.6. *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group. Then there exists $N < \infty$, depending on the group G but not the HHG structure, such that the following holds. Let $A \leqslant G$ be a virtually abelian subgroup. Then there is a free abelian subgroup $A_1 \leqslant A$ such that $[A : A_1] \leqslant N$ and $C_G(A_1)$ is hierarchically quasiconvex with respect to any HHG structure on G, and in particular, $C_G(A_1)$ admits an HHS structure where the left multiplication action of $C_G(A_1)$ on itself is an action by HHS automorphisms.*

Motivation for Corollary 1.6 comes from the fact that a given finitely generated subgroup $H \leqslant G$ may by HQC or not, depending on how we choose the HHG structure, which, in general, can be done in many ways; see [Man24] and also [AHPZ25, Thm. 5.15, Rem. 5.16].

Passing to the finite-index subgroup $A_1 \leqslant A$ in Corollary 1.6 is needed in order to rule out the possibility that $aV \perp V$ for certain $V \in \mathfrak{S}$, where $a \in A$. In particular, if $(G, \mathfrak{S})$ has the global property that elements of $\mathfrak{S}$ are not orthogonal to their G–translates, then we get hierarchical quasiconvexity of centralisers. An emblematic example of such a situation is when G is the fundamental group of a compact special cube complex, with the HHG structure from [BHS17a].

Corollary 1.7. *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group and suppose that $gU \not\perp U$ for all $g \in G$ and $U \in \mathfrak{S}$. Let A be a free abelian subgroup. Then $C_G(A)$ is hierarchically quasiconvex.*

This is proved in Section 7, as part of the proof of Corollary 7.6. The hypothesis that elements of $\mathfrak{S}$ are not orthogonal to their G–translates holds in a finite-index subgroup whenever G is *colourable* in the sense of [DMS23]; while there are non-colourable HHGs [Hag23a], many of the natural examples of HHGs, like mapping class groups [BBF15] and compact special groups [BHMS24a], are colourable.

Remark 1.8. The preceding corollary is related to work of Genevois [Gen22, Thm. 5.1, Lem. 5.2], which shows that, whenever G acts geometrically on a cube complex X, then, up to subdividing X once, for each infinite order $g \in G$, there is an ℓ_1–isometric subcomplex $Min(g)$ of X — a connected median subalgebra — stabilised cocompactly by $C_G(\langle g \rangle)$.

[3] There is a limited analogue of the Thurston classification for general HHGs, which does play an important role in this paper, that goes back to the classification of HHS automorphisms/"coarse rank rigidity" in [DHS17] and has its most modern form in [PS23, Thm. 5.1]. However, this is a statement about an action on a hierarchical structure, and doesn't rely on anything specific to surfaces, or even directly imply the Thurston classification in the MCG case.

Now, if G also acts cospecially, then, as shown in [Gen21], $Min(g)$ coincides with the *stable* minset $SMin(g)$ which, by [Gen21, Rem. 3.7], is convex, giving cubical convexity (and hence hierarchical quasiconvexity) of centralisers. See also [HP20, Prop. 9.6]. ✽

We finally discuss the structure of commensurators of virtually abelian subgroups in an HHG $(G, \mathfrak{S})$, inspired by the corresponding discussion of commensurators of abelian subgroups of CAT(0) groups from [HP20]. We prove:

Corollary 7.7. *Let $(G, \mathfrak{S})$ be an HHG and let $A \leqslant G$ be a virtually abelian subgroup. Let $\mathrm{Comm}_G(A)$ be the commensurator of A in G. Let $K \leqslant \mathrm{Comm}_G(A)$ be a finitely generated subgroup. Then $K \leqslant N_G(A')$, where $A' \leqslant A$ is a finite-index subgroup.*

An observation in [HP20] about complete square complexes (CSCs) shows that this result is sharp, in that one cannot in general replace the finitely generated subgroup K with the entire commensurator $\mathrm{Comm}_G(A)$. Now, in [HP20, Cor. 5.10], Huang–Prytuła show that, if G is compact special (in particular, both a CAT(0) group and an HHG), $\mathrm{Comm}_G(A)$ is finitely generated and coincides with $N_G(A')$ for some finite-index $A' \leqslant A$.

When G is a mapping class group, [JRLASSn24, Thm. 1.1, Rem. 1.2] shows that, if $A \leqslant G$ is a virtually abelian subgroup, then $\mathrm{Comm}_G(A)$ coincides with $N_G(A')$, where A' is a finite-index with A. This follows from Corollary 7.7 once one knows that $\mathrm{Comm}_G(A)$ is finitely generated, which is shown for mapping class groups in Corollary 4.6 in loc. cit., but which does not hold for general HHGs.

In various recent papers, some attention is given to the possibility of a theory of "algebraically well-behaved" HHGs, encompassing compact special groups and mapping class groups. Roughly speaking, such a theory should involve imposing conditions ensuring that various HQC subgroups are separable, and that coarse products of subgroups must arise from a certain amount of commutation. For instance, the idea of requiring separability of product region subgroups, motivated by mapping class groups [LM07], is explored in [HP22, Prop. 3.2]. The correct commutation condition may turn out to vary according to the exact applications one wants, but the notions of "weakly commutative" [OP25, Defn. 3.4], "decomposable" [DMS25] or "algebraic" [CRHK24] HHGs, or the conditions from [AB23] are all candidate conditions aimed at excluding colourable HHGs whose algebraic properties do not reflect their geometry in straightforward ways, like irreducible CSC groups. We wonder if imposing the condition that abelian subgroups have finitely generated commensurators is a useful move in this direction.

1.1. **Quasiflat closing.** The famous Flat Closing Problem for CAT(0) spaces asked whether a group G acting geometrically on a CAT(0) space X with an isometrically embedded $\mathbb{E}^2$ must contain a $\mathbb{Z}^2$ subgroup. This was spectacularly resolved by the counterexample provided by Martelli in the recent paper [Mar25].

In contrast, we prove the following positive result for hierarchically hyperbolic groups:

Theorem 7.14. *Let $(G, \mathfrak{S})$ be an HHG with rank ν. Then there exists $\mathbb{Z}^\nu \leqslant G$.*

A special case of this, Theorem 7.13, says that any HHG is either hyperbolic or contains a $\mathbb{Z}^2$ subgroup. In Section 7.3, we first present proof of the latter theorem, to illustrate the ideas avoiding some complications, and then generalise the argument to prove Theorem 7.14.

For CAT(0) cube complexes, Theorem 7.13 and the results in [BHS17a] immediately yield:

Corollary 1.9. *Flat closing holds for CAT(0) cube complexes which admit a factor system.*

It is unknown whether flat closing holds for general cocompactly cubulated groups (not all of which admit factor systems [She23]), but there were some previously known positive results. For example, it is well known that if G is virtually compact special in the sense of

[HW08], then G is hyperbolic or contains $\mathbb{Z}^2$. Work of Sageev–Wise [SW11] yields the same conclusion if G acts geometrically on a CAT(0) cube complex satisfying a condition that limits intersections between certain sets of hyperplane stabilisers (and is therefore somewhat similar in spirit to the existence of a factor system).

Theorem 7.14 follows from Corollary 7.8 together with an argument that is elementary, in the sense that it uses only the core concepts of the theory of HHSes. The key observation is that Corollary 7.8 reduces the problem to that of finding some $g \in G$ such that there are at least two distinct $U \in \mathfrak{S}$ for which $\pi_U(\langle g \rangle)$ is unbounded. In the case of a cube complex with a factor system, this can also be deduced from [WW17]. The factor system case also follows by combining [CS11, Cor. D] and [HS20, Prop. 2.7]; however, it appears that this statement does not appear elsewhere in the literature.

1.2. **Sketch of the proof.** Here is a summary of the proof of Theorem 1.3.

As in the proof of [HRSS25, Prop. 2.17], we first find pairwise-orthogonal $U_1, \dots, U_n \in \mathfrak{S}$ such that $\{U_1, \dots, U_n\}$ is A–invariant, and any $W \in \mathfrak{S}$ with π_W unbounded on A–orbits must satisfy $W \sqsubseteq U_i$ for some i, using [PS23, Thm. 5.1]. This allows us to use tools from [BHS19], like the realisation theorem and distance formula, along with a modified version of the "factored space" construction from [BHS17b], to reduce to the case where $X = H \times \check{X}$ (coarsely), where H is an HHS whose index set contains the set $\mathfrak{U}$ of all U nested in some U_i, and $\check{X}$ is an HHS where A acts by HHS automorphisms with bounded orbits. Moreover, letting $A' \leqslant A$ be a bounded-index subgroup fixing each U_i, we observe that $\mathcal{C}U_i$ contains an A'–invariant uniform quasiline γ_{U_i}. In fact, the subgroup A' stabilises two points $p^{\pm}$ in the HHS boundary from [DHS17], and γ_{U_i} joins the two points $p^{\pm}_{U_i} \in \partial\mathcal{C}U_i$ to which $p^{\pm}$ project.

For $U \in \mathfrak{U}$, either $\pi_U(X)$ is uniformly bounded, or γ_{U_i} projects to a (bounded, but not uniformly) uniform quasigeodesic γ_U. By passing to a suitable uniformly HQC subspace, we can assume that $\pi_U(H)$ is either uniformly bounded, or uniformly coarsely coincides with γ_U, for all U. These arguments are collected in Proposition 5.6.

In Proposition 4.1, we use the fact that $\check{X}$ is coarsely dense in its injective hull, as shown in [HHP23], together with Lang's fixed-point theorem for injective spaces [Lan13] and the fact that A has bounded orbits in $\check{X}$, to choose a point $\check{x} \in \check{X}$ whose A–orbit has uniformly bounded diameter. This reduces the problem to finding the required A–invariant, uniformly EHQC, uniform quasiflat in H.

This is done using Theorem 3.1, which is a variant of Durham's cubical model theorem, and in fact we borrow our strategy from [Dur23]. The latter is itself a considerable generalisation and strengthening of the cubical approximation theorem from [BHS21] (which was reproved by Bowditch by different means in [Bow18a]), and strengthens Durham–Zalloum's cubical model theorem for HHS boundary points [DZ22]. (See [Zal23] for a survey of the various cubical modelling tools available in HHSs, and [DMS23] for another strengthened form of cubical approximation for colourable HHGs.)

Given the inputs to Theorem 3.1, Durham's theorem would provide (depending in principle on the choice $z_0 \in H$ of basepoint) a uniform quasi-isometry from a CAT(0) cube complex Q to the HHS H, preserving (coarse) medians up to uniformly bounded error. Durham also obtains a form of equivariance, but it is not quite what we need. We restrict ourselves to a more limited setting than in Durham's work, because we are concerned only with the case where H is the hull of a pair of boundary points (Durham allows arbitrary finite sets), which lets us dispense with basepoints and obtain a complete, connected, finite-rank median space Q and a uniformly quasi-median quasi-isometry $H \to Q$, as in Durham's theorem, but with the added feature that A acts on Q by isometries and the map to Q is A–equivariant. From there, a result of Bowditch [Bow16] yields a compatible CAT(0) metric on Q to which we can apply the classical flat torus theorem to obtain the desired EHQC quasiflat in H.

To prove the more detailed version of our theorem (Theorem 6.1), we consider the set of all A–invariant quasi-flats obtained in this way. The resulting quasi-product structure $Y \times C \times \mathbb{E}^n$ comes from the fact that the fixed point set of A in the injective hull of $\check{X}$ is itself injective, and that the CAT(0) flat torus theorem provides an A–invariant subspace $C \times \mathbb{E}^n \subseteq Q$ such that C is closed and convex (and thus CAT(0)).

1.3. **Outline of the paper.** Section 2 contains background on hierarchically hyperbolic spaces, as well as a few miscellaneous facts needed later in the paper. Section 3 is devoted to the proof of Theorem 3.1, on median models of hulls. Section 4 deals with bounded actions on HHSs, yielding Proposition 4.1. In Section 5, we prove Proposition 5.6, providing the necessary canonical A–invariant HQC and EHQC subspaces of X. In Section 6, we combine these ingredients to prove our main theorems, Theorem 6.1 and Theorem 6.5, and we apply these to obtain the various corollaries in Section 7.

There is also Appendix A, where we restate several existing results about hierarchical hyperbolicity, making explicit how constants produced by these statements depend on the inputs. In no such case do we present a new proof; we just extract from the proofs in the literature the required information about constants. Typically, we require some output constant to depend only on the parameters of the HHS, and obtaining this is a matter of tracing through an existing proof to get a quantitative version of the statement we need.

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2. Background on hierarchically hyperbolic spaces

Fix a hierarchically hyperbolic space (HHS) $(X, \mathfrak{S})$, as defined in [BHS19, Defn. 1.1]. We adopt the notation from there, some of which we recall here.

For each $U \in \mathfrak{S}$, let $\pi_U : X \to \mathcal{C}U$ be the associated projection to the hyperbolic space $\mathcal{C}U$. Whenever $U, V \in \mathcal{C}U$ satisfy $U \subsetneq V$ or $U \pitchfork V$, the uniformly bounded set in $\mathcal{C}V$ associated to U is denoted ρ^U_V, and similarly, if $U \subsetneq V$, there is a map $\rho^V_U : \mathcal{C}V \to \mathcal{C}U$. Let d be the metric on X, which is assumed to be a quasigeodesic space. For each $V \in \mathfrak{S}$, let d_V be the (hyperbolic geodesic) metric on $\mathcal{C}U$. Following the usual convention, given $x, y \in X$, we write $d_V(x, y)$ to mean $d_V(\pi_V(x), \pi_V(y))$.

2.1. **HHS boundary and boundary point projections.** We recall the definition of underlying set of the *HHS boundary* of $(X, \mathfrak{S})$ from [DHS17]. We will not need the topology.

Definition 2.1. The *HHS boundary* of an HHS $(X, \mathfrak{S})$ is a set ∂X of points $\lambda \in \partial X$, each of which is determined by the following data:

(1) a pairwise orthogonal set of domains $\mathrm{supp}(\lambda) \subseteq \mathfrak{S}$, called the *support* of λ;
(2) a point in the Gromov boundary $\lambda^U \in \partial\mathcal{C}U$ for each $U \in \mathrm{supp}(\lambda)$;
(3) a collection of positive real numbers $\{a_U : U \in \mathrm{supp}(\lambda)\}$ such that $\sum_{U \in \mathrm{supp}(\lambda)} a_u = 1$.

Set $\mathrm{supp}^{\perp}(\lambda) := \{V \in \mathfrak{S} : V \perp U \text{ for all } U \in \mathrm{supp}(\lambda)\}$. ✽

Notation 2.2. Given $\delta > 0$, denote by Morse_δ the function provided by the Morse lemma for δ–hyperbolic geodesic spaces, i.e. for any ϵ_1, ϵ_2, any (ϵ_1, ϵ_2)–quasigeodesic in such a space is at Hausdorff distance at most $\mathsf{Morse}_\delta(\epsilon_1, \epsilon_2)$ from the geodesic joining its endpoints. ✽

The projection maps π_U can be extended to the boundary as follows. This process is canonical unless $U \in \mathrm{supp}^\perp$, in which case it requires a choice of basepoint in X.

Let $\delta \geqslant 0$ be the hyperbolicity constant (so $\mathcal{C}U$ is δ-hyperbolic for all $U \in \mathfrak{S}$) and let $\mathcal{E}$ be the constant from the bounded geodesic image axiom ([BHS19, Defn. 1.1.(7)]). Let $\lambda \in \partial X$, let $U \in \mathrm{supp}(\lambda)$, and let $V \in \mathfrak{U}$ be such that $V \subsetneq U$. Denote by $\lambda_U^V \subseteq \mathcal{C}U$ the union of all $(1, 20\delta)$-quasi-geodesic rays γ in $\mathcal{C}U$ in the equivalence class corresponding to the boundary point λ^U such that $\gamma \cap \mathcal{N}^{\mathcal{C}U}_{\mathcal{E}+\mathsf{Morse}_\delta(1,20\delta)}(\rho_U^V) = \varnothing$.

Definition 2.3 (Projections of boundary points)**.** Let $\lambda \in \partial X$ and let $x_0 \in X$. For each $U \in \mathfrak{S}$, define the *projection based at* x_0, denoted $\pi_{U,x_0}(\lambda) \in 2^{\mathcal{C}U} \cup \partial\mathcal{C}U$, as follows:

- if $U \in \mathrm{supp}^\perp(\lambda)$ then let $\pi_{U,x_0}(\lambda) := \pi_U(x_0)$;
- if $U \in \mathrm{supp}(\lambda)$ then let $\pi_{U,x_0}(\lambda) := \lambda^U$;
- if $U \notin \mathrm{supp}(\lambda)$ and $U \subsetneq V$ for some $V \in \mathrm{supp}(\lambda)$ then let $\pi_{U,x_0}(\lambda) := \rho_U^V(\lambda_V^U)$. Also set $\rho_U^V(\pi_V(\lambda, x_0)) := \pi_U(\lambda, x_0)$.
- If none of the above hold, then the set $\mathcal{V} := \{V \in \mathrm{supp}(\lambda) : V \subsetneq U \text{ or } V \pitchfork U\}$ is non-empty and we define $\pi_{U,x_0}(\lambda) := \bigcup_{V \in \mathcal{V}} \rho_U^V$.

If $V \in \mathfrak{S} - \mathrm{supp}^\perp(\lambda)$, then $\pi_{V,x_0}(\lambda)$ is independent of x_0, so we can write $\pi_V(\lambda) := \pi_{V,x_0}(\lambda)$. ✽

2.2. **HHS parameters.** Our main result asserts that certain "output" quantities depend only on the quantities in the definition of an HHS ([BHS19, Defn. 1.1]), namely:

(1) q is the quasigeodesic constant for X, i.e. any two points are joined by a (q, q)–quasigeodesic.
(2) δ is the hyperbolicity constant, i.e. $\mathcal{C}U$ is δ–hyperbolic for all $U \in \mathfrak{S}$.
(3) Constants K and ξ: for all $U \in \mathfrak{S}$, the coarse map π_U is (K, K)–coarsely lipschitz and has K–quasiconvex image, while $\pi_U(x)$ has diameter at most ξ for all $x \in X$. Also, $\mathsf{diam}(\rho_V^U) \leqslant \xi$ whenever $U \subsetneq V$ or $U \pitchfork V$.
(4) There is a constant κ_0 such that all of the inequalities in [BHS19, Defn. 1.1.(4)] (the transversality/consistency axiom) use the upper bound κ_0.
(5) The *complexity* $\chi \in \mathbb{N}$ is the maximum length of a $\subsetneq$–chain in $\mathfrak{S}$.
(6) The large link axiom [BHS19, Defn. 1.1.(6)], bounded geodesic image axiom [BHS19, Defn. 1.1.(7)] and partial realisation axiom ([BHS19, Defn. 1.1.(8)]) each introduce one additional constant, denoted $\lambda, \mathcal{E}$ and α respectively.
(7) The uniqueness axiom ([BHS19, Defn. 1.1.(9)]) provides a function $\theta_u : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that $d(x, y) \leqslant \theta_u(\sup_{V \in \mathfrak{S}} d_V(x, y))$ for all $x, y \in X$.

Using the definition of boundary projections in Section 2.1, it is straightforward to prove:

Lemma 2.4. *If $\lambda \in \partial X$, $x_0 \in X$ and $U \in \mathfrak{S} - \mathrm{supp}(\lambda)$, then $\mathsf{diam}(\pi_{U,x_0}(\lambda)) \leqslant \xi'$, where ξ' is a function of ξ, α and $\mathcal{E}$.*

Proof. Let $U \in \mathfrak{S} - \mathrm{supp}(\lambda)$. There are three cases.

First suppose that $U \perp V$ for all $V \in \mathrm{supp}(\lambda)$. Then $\mathsf{diam}(\pi_{U,x_0}(\lambda)) = \mathsf{diam}(\pi_U(x_0)) \leqslant \xi$.

Second, suppose that $U \pitchfork V$ or $V \subsetneq U$ for some $V \in \mathrm{supp}(\lambda)$, i.e. the set $\mathcal{V}$ from Definition 2.3 is nonempty. By definition, $\pi_{U,x_0}(\lambda) = \bigcup_{V \in \mathcal{V}} \rho_U^V$. Since $V \perp V'$ for all distinct $V, V' \in \mathcal{V}$, we have $d_U(\rho_U^V, \rho_U^{V'}) \leqslant 2\alpha$, so $\mathsf{diam}(\pi_{U,x_0}(\lambda)) \leqslant 2(\alpha + \xi)$.

Third, suppose that $U \subsetneq V$ for some $V \in \mathrm{supp}(\lambda)$. Recall that $\lambda_V^U \subseteq \mathcal{C}V$ is the set of $(1, 20\delta)$–quasigeodesic rays γ representing the boundary point λ^V and avoiding the neighbourhood of ρ_V^U of radius $\mathcal{E}_0 := \mathcal{E} + \mathsf{Morse}_\delta(1, 20\delta)$. Recall that in this case, $\pi_{U,x_0}(\lambda)$ is defined to be $\rho_U^V(\lambda_V^U)$, whose diameter we need to bound. Let γ be a ray in $\mathcal{C}V$ as above. Let $a, b \in \gamma$. Let β be a geodesic in $\mathcal{C}V$ from a to b. Since β lies in the $\mathsf{Morse}_\delta(1, 20\delta)$–neighbourhood of γ, it cannot come $\mathcal{E}$–close to ρ_V^U, so by the bounded geodesic image axiom

([BHS19, Defn. 1.1.(7)]), $d_U(\rho^V_U(a), \rho^V_U(b)) \leqslant \mathcal{E}$. Since this works for arbitrary a, b, we have $\mathsf{diam}(\pi_U(\gamma)) \leqslant \mathcal{E}$. Now let γ' be some other $(1, 20E)$–quasigeodesic ray in $\mathcal{C}V$ avoiding $\mathcal{N}_{\mathsf{Morse}_\delta(1,20\delta)+\mathcal{E}}(\rho^U_V)$ and representing λ^V. Since ρ^U_V is bounded and γ, γ' are at finite Hausdorff distance, there exist $a \in \gamma, a' \in \gamma'$ such that every geodesic in $\mathcal{C}V$ from a to a' avoids $\mathcal{N}_{\mathcal{E}}(\rho^U_V)$. Hence $d(\rho^V_U(\gamma), \rho^V_U(\gamma')) \leqslant \mathcal{E}$, by another application of bounded geodesic image. Thus $\mathsf{diam}(\rho^V_U(\lambda^U_V)) \leqslant 3\mathcal{E}$.

Combining the three cases gives $\mathsf{diam}(\pi_{U,x_0}(\lambda)) \leqslant 2(\alpha + \xi) + 3\mathcal{E}$. □

Definition 2.5 (HHS constant)**.** Fix $E \geqslant \max\{q, K, \delta, \xi, \xi', \kappa_0, \alpha, \mathcal{E}\}$; E is the *HHS constant.* ✽

Remark 2.6. Definition 2.5 is almost identical to what is usually referred to as the HHS constant (see e.g [BHS19, Remark 1.6]) except for the condition that $E \geqslant \xi'$ which accounts for the boundary projections. As usual, since all of the statements in [BHS19, Defn. 1.1] involving the quantities used to define E remain true when the quantity in question is replaced by E, we henceforth work with E instead of κ_0, α, etc. ✽

Definition 2.7 (Complexity)**.** We denote the complexity of $(X, \mathfrak{S})$ by χ. Note that it depends only on the poset $(\mathfrak{S}, \sqsubseteq)$. ✽

Definition 2.8 (HHS parameters)**.** The *HHS parameters* of $(X, \mathfrak{S})$ are: the HHS constant E, the complexity χ, and the uniqueness function θ_u. ✽

When we say that some quantity C depends only on the HHS parameters, we mean that C is a function of E, χ, and θ_u. For brevity, when working in a fixed HHS, we say that a quantity is *uniform* to mean that it depends only on the HHS parameters.

Definition 2.9 (Adjusting the boundary projections)**.** In the interest of only using the constant E from now on, we make a small change to the definition of the boundary projections. If $\lambda \in \partial X$ and $V \subsetneq U$ for some $U \in \mathrm{supp}(\lambda)$ then let $\lambda'^V_U := \lambda^V_U - \mathcal{N}_{10E}(\rho^V_U)$. Define $\pi_U(\lambda)' := \rho^U_V(\lambda'^V_U)$. Then $\pi_U(\lambda)' \subseteq \pi_U(\lambda)$ so its diameter is still bounded by E. We replace λ^V_U by λ'^V_U and $\pi_U(\lambda)$ by $\pi_U(\lambda)'$. ✽

2.3. **HHS automorphisms and HHGs.** We work with the following notion of *HHS automorphisms* introduced in [PS23, Sec. 2], which is simpler to work with than the original formulation (and no less general, by [DHS20, Sec. 2.1]).

Definition 2.10 (HHS automorphism)**.** Let $(X, \mathfrak{S})$ be an HHS. A group H acts on $(X, \mathfrak{S})$ by *HHS automorphisms* if there is an action $H \to Sym(X)$, an action $H \to Sym(\mathfrak{S})$ preserving the relations $\sqsubseteq, \perp, \pitchfork$, and, for each $U \in \mathfrak{S}$ and each $h \in H$, an isometry $h : \mathcal{C}U \to \mathcal{C}hU$ such that all of the following hold:

(1) If $U \in \mathfrak{S}$ and $g, h \in H$, then the isometry $gh : \mathcal{C}U \to \mathcal{C}ghU$ is the composition of $g : \mathcal{C}hU \to \mathcal{C}ghU$ with $h : \mathcal{C}U \to \mathcal{C}hU$.
(2) $\pi_{gU}(gx) = g(\pi_U(x))$ for all $x \in X, U \in \mathfrak{S}, g \in H$.
(3) $\rho^{gU}_{gV} = g(\rho^U_V)$ whenever $U \subsetneq V$ or $U \pitchfork V$. ✽

If H acts by HHS automorphisms, the distance formula (Theorem A.1) implies that each $h \in H$ acts on X as a quasi-isometry with constants depending on the HHS parameters only, but in this paper, it will always suffice to work with groups H of HHS automorphisms with the additional property that the H–action on (X, d) is isometric. A special case is:

Definition 2.11 (HHG)**.** Let G be a group and let d be a finitely generated word metric on G. Suppose that (G, d) admits an HHS structure $(G, \mathfrak{S})$ on which G acts by HHS automorphisms in such a way that the associated G–action on the space (G, d) is the left-multiplication action, and the G–action on $\mathfrak{S}$ is cofinite. Then $(G, \mathfrak{S})$ is a *hierarchically* *hyperbolic group (HHG) structure* for G, and G is a *hierarchically hyperbolic group (HHG)* if it has an HHG structure. We say that the pair $(G, \mathfrak{S})$ is *an HHG* in this situation. ✽

Any HHS automorphism g extends to the HHS boundary: if $\lambda \in \partial X$ is as in Definition 2.1 then $g\lambda \in \partial X$ is determined by the support $g\,\mathrm{supp}(\lambda)$, the boundary points $g\lambda^U \in \partial \mathcal{C}gU$ for all $gU \in g\,\mathrm{supp}(\lambda)$ and the constants $a_{gU} := a_U$ for all $gU \in g\,\mathrm{supp}(\lambda)$.

Lemma 2.12 (Equivariant downward ρ maps)**.** *Up to uniformly modifying the HHS parameters, we can assume that*

- $\pi_U(x) \subseteq \rho_U^V(\pi_V(x))$ *whenever* $x \in X$ *and* $U \subsetneq V$*.*
- *if* g *is an HHS automorphism of* X *and* $U, V \in \mathfrak{S}$ *satisfy* $U \subsetneq V$*, then*
$$\rho_{gU}^{gV}(gz) = g\rho_U^V(z)$$
for all $z \in \mathcal{C}V - \mathcal{N}_{10E}(\rho_U^V)$*.*

Proof. Let $x \in X$ and let $U, V \in \mathfrak{S}$ satisfy $U \subsetneq V$. Let $z \in \mathcal{C}V$, and suppose that $d_V(z, \rho_V^U) \geqslant 10E$. Let $x_z := \{y \in X : d_V(y, z) \leqslant E\}$, which is non-empty by the coarse surjectivity of π_V. Let $\varrho_U^V(z) = \bigcup_{x \in x_z} \pi_U(x)$. The consistency and bounded geodesic image axioms ([BHS19, Defn. 1.1.(4),(6)]) imply that $\varrho_U^V(z)$ has diameter bounded in terms of E, and lies uniformly close (in terms of E) to $\rho_U^V(z)$. In particular, we can (and do) replace ρ_U^V by ϱ_U^V in the HHS structure $(X, \mathfrak{S})$ and still have an HHS whose parameters are bounded in terms of the original ones. (For $z \in \mathcal{C}V$ with $d_V(z, \rho_V^U) < 10E$, let $\varrho_U^V(z) = \rho_U^V(z)$.)

Let g be an HHS automorphism and let $z \in \mathcal{C}V$ satisfy $d_V(z, \rho_V^U) \geqslant 10E$. Definition 2.10 gives $d_{gV}(gz, \rho_{gV}^{gU}) \geqslant 10E$ and $gx_z = x_{gz}$, so $\rho_{gU}^{gV}(gz) = \pi_{gU}(gx_z) = g\pi_U(x_z) = g\rho_U^V(z)$. □

We will henceforth assume that the downwards ρ maps are defined as in the proof of Lemma 2.12 and thus satisfy its conclusions. As a result, the boundary projections are also equivariant with respect to HHS automorphisms:

Lemma 2.13. *Let* $\lambda \in \partial X$ *be a point in the HHS boundary of* X*, let* g *be an HHS automorphism and let* $U \in \mathfrak{S}$*. Then* $g\pi_{U,x_0}(\lambda) = \pi_{gU,gx_0}(g\lambda)$ *for all* $x_0 \in X$*.*

Proof. The only case which is not covered by Definition 2.10 is if $U \subsetneq V$ for some (necessarily unique) $V \in \mathrm{supp}(U)$. In this case $\pi_U(\lambda) = \rho_U^V(\lambda_V^U)$, $\pi_{gU} = \rho_{gU}^{gV}(\lambda_{gV}^{gU})$ and $\lambda_V^U \subseteq \mathcal{C}V - \mathcal{N}_{10E}(\rho_V^U)$, $\lambda_{gV}^{gU} \subseteq \mathcal{C}gV - \mathcal{N}_{10E}(\rho_{gV}^{gU})$. By Lemma 2.12 it follows that $g\pi_U(\lambda) = \pi_{gU}(g\lambda)$. □

2.4. **Realisation.** We recall from [BHS19] the definition of a *consistent tuple*, and the *realisation theorem* that, roughly speaking, characterises the image of $\prod_U \pi_U : X \to \prod_U \mathcal{C}U$ as the "solution set" of the consistency inequalities. The following is [BHS19, Defn. 1.17]:

Definition 2.14 (Consistent tuple)**.** Let $\kappa \geqslant 0$. Let $\vec{b} = (b_U)_{U \in \mathfrak{S}} \in \prod_{U \in \mathfrak{S}} 2^{\mathcal{C}U}$. Then $\vec{b}$ is *κ–admissible* if $\mathsf{diam}(b_U) \leqslant \kappa$ and $d_U(b_U, \pi_U(X)) \leqslant \kappa$ for all $U \in \mathfrak{S}$. If $\vec{b}$ is κ–admissible, then it is *κ–consistent* if

- $\min\{d_U(\rho_U^V, b_U), d_V(\rho_V^U, b_V)\} \leqslant \kappa$ whenever $U \pitchfork V$, and
- $\min\{d_V(\rho_V^U, b_V), \mathsf{diam}(\rho_U^V(b_V) \cup b_U)\} \leqslant \kappa$ whenever $U \subsetneq V$. ✽

Next is the realisation theorem, which is Theorem 3.1 in [BHS19]. Following [CRHK24, Thm. 12.5], we have restated it to make explicit what the quantities involved depend on; as noted in [CRHK24], the constants in question are easy to extract from the proof in [BHS19].

Theorem 2.15 (Realisation)**.** *Let* $(\mathcal{X}, \mathfrak{S})$ *be a hierarchically hyperbolic space. Then there exists* r_0*, depending only on the HHS parameters, such that the following holds. Let* $\kappa \geqslant 1$ *and let* $(b_V)_{V \in \mathfrak{S}} \in \prod_{V \in \mathfrak{S}} \mathcal{C}V$ *be a* κ*–consistent,* κ*–admissible tuple. Then there exists* $x \in \mathcal{X}$ *such that* $d_V(x, b_V) \leqslant r_0\kappa$ *for all* $V \in \mathfrak{S}$*.*

2.5. **Coarse median.** We use the notion of a *coarse median space* from [Bow13, Sec. 8], although we almost always work with special cases — median spaces or HHSes — rather than the general definition. See [NWZ21] for more background on *coarse median algebras/spaces.*

Definition 2.16 (Coarse median space)**.** A metric space (Ω, d) is *coarse median* with *parameter* $\kappa : \mathbb{Z}_{\geqslant 0} \to [0, \infty)$ if there is a map $\mu : \Omega^3 \to \Omega$ (the *coarse median*) such that:

(C1) We have $d(\mu(x, y, z), \mu(x', y', z')) \leqslant \kappa(0)\,(d(x, x') + d(y, y') + d(z, z')) + \kappa(0)$ for all $x, y, z, x', y', z' \in \Omega$.

(C2) For all $n \in \mathbb{N}$ and all $A \subseteq \Omega$ with $|A| \leqslant n$, there is a finite median algebra (M, η) and maps $f : M \to \Omega$ and $\bar{f} : A \to M$ such that $d(\mu(f(x), f(y), f(z)), f(\eta(x, y, z))) \leqslant \kappa(n)$ for all $x, y, z \in M$, and $d(a, f\bar{f}(a)) \leqslant \kappa(n)$ for all $a \in A$. ✽

Next is the coarse median on an HHS X, from [BHS19, Sec. 7] (or [CRHK24, Sec. 14]).

Construction 2.17 (HHS coarse median)**.** Let $x, y, z \in X$. For each $U \in \mathfrak{S}$, let m_U be the set of $p \in \mathcal{C}U$ such that $d_U(p, \gamma) \leqslant 10E$ for any geodesic γ of $\mathcal{C}U$ with endpoints in $\pi_U(x) \cup \pi_U(y) \cup \pi_U(z)$. Let $(b_U)_{U \in \mathfrak{S}}$ be a tuple with $b_U \in m_U$ for all U. By [BHS19, Lem. 2.6], there exists κ such that any such $(b_U)_{U \in \mathfrak{S}}$ is κ–consistent, and the proof of the same lemma shows we can take $\kappa = 100E$. Hence, the realisation theorem (Theorem 2.15) provides a point $m \in X$ such that $d_U(m, b_U) \leqslant 100Er_0$, where r_0 is the constant, depending only on the HHS parameters, from Theorem 2.15.

Let $\mu(x, y, z)$ be the set of all $m \in X$ such that $d(m, m_U) \leqslant 100Er_0$ for all $U \in \mathfrak{S}$. The above discussion shows that $\mu(x, y, z) \neq \varnothing$. Now, if $m, m' \in \mu(x, y, z)$, then for all U, we have $d_U(m, m') \leqslant 200E(r_0 + 1)$, so $d(m, m') \leqslant \theta_u(200E(r_0 + 1))$, where θ_u is the uniqueness function (an HHS parameter). Thus $\mathsf{diam}(\mu(x, y, z))$ is bounded, independently of x, y, z, in terms of the HHS parameters only. Hence μ is a well-defined map sending points in X^3 to uniformly bounded sets in X.

In summary, Construction 2.17 provides constants C_0, C_1, depending only on the HHS parameters, and a map $\mu : X^3 \to 2^X$ satisfying both of the following conditions:

- $\mathsf{diam}(\mu(x, y, z)) \leqslant C_0$ for all $x, y, z \in X$.
- For all $U \in \mathfrak{S}$ and $x, y, z \in X$, the following holds. Let $a, b \in \{x, y, z\}$ be distinct. Then $d_U(\mu(x, y, z), \gamma) \leqslant C_1$, where C_1 is any $\mathcal{C}U$–geodesic joining a point in $\pi_U(a)$ to a point in $\pi_U(b)$.

If a group G acts on $(X, \mathfrak{S})$ by HHS automorphisms, then the coarse map μ is G–equivariant (for the diagonal action on X^3 and the action on 2^X induced by the G–action on X).

In [CRHK24, Sec. 14], it is observed that if $(G, \mathfrak{S})$ is an HHG then we can, moreover, assume that μ is a map from $X^3 \to X$ (rather than a coarse map) by equivariantly choosing points in the bounded sets $\mu(x, y, z)$ defined above. This can be done, more generally, as long as G acts on $(X, \mathfrak{S})$ by HHS automorphisms in such a way that the G–action on X is free. Later, we will typically reduce to the case of free actions, and in such situations we will assume that μ is an equivariant *map*, not just an equivariant coarse map.

Remark 2.18. That μ satisfies Definition 2.16 follows from the cubical approximation theorem explained in Section A.2; see Corollary A.3. ✽

Definition 2.19 (Quasimedian map)**.** Let Λ, Ω be coarse median spaces with coarse medians λ, ω respectively; let d be the metric on Ω. Let $\kappa \geqslant 0$. A map $f : \Lambda \to \Omega$ is κ*–quasimedian* if $d(\omega(f(a), f(b), f(c)), f(\lambda(a, b, c))) \leqslant \kappa$ for all $a, b, c \in \Lambda$. ✽

The next two definitions help to interpret HHS notions in terms of the coarse median.

Definition 2.20 (Median quasiconvexity)**.** Let Ω be a coarse median space with coarse median ω and metric d. Given $\kappa \geqslant 0$, a subset $Y \subseteq \Omega$ is κ*–median-quasiconvex* if $d(\omega(y, y', x), Y) \leqslant \kappa$ for all $y, y' \in Y$ and $x \in \Omega$. ✽

Definition 2.21 (Coarse subalgebra)**.** Let Ω be a coarse median space with coarse median ω and metric d. Given $\kappa \geqslant 0$, a *κ–coarse median subalgebra* of Ω is a subset $Y \subseteq \Omega$ such that $d(\omega(y, y', y''), Y) \leqslant \kappa$ for all $y, y', y'' \in Y$. ✽

2.6. **Hierarchical quasiconvexity and related notions.** We recall the notion of hierarchical quasiconvexity from [BHS19, Sec. 5]:

Definition 2.22. Given a function $\kappa : [0, \infty) \to [0, \infty)$, a subset $Y \subseteq X$ is *κ–hierarchically quasiconvex (HQC)* if both of the following hold:

- $\pi_U(Y)$ is $\kappa(0)$–quasiconvex in the E–hyperbolic space $\mathcal{C}U$ for all $U \in \mathfrak{S}$;
- for all $r \geqslant 0$, if $x \in X$ satisfies $d_U(x, Y) \leqslant r$ for all $U \in \mathfrak{S}$, then $d(x, Y) \leqslant \kappa(r)$. ✽

We next have [RST23, Prop. 5.11], relating hierarchical and coarse median quasiconvexity:

Proposition 2.23. *Let $(X, \mathfrak{S})$ be an HHS and let $Y \subseteq X$. Then both of the following hold:*

(1) For all $\kappa : [0, \infty) \to [0, \infty)$, there exists $C \geqslant 0$, depending only on κ and the HHS parameters, such that, if Y is κ–hierarchically quasiconvex, then Y is C–median quasiconvex.

(2) For all $C \geqslant 0$, there exists $\kappa : [0, \infty) \to [0, \infty)$, depending only on C and the HHS parameters, such that, if Y is C–median quasiconvex, then it is κ–hierarchically quasiconvex.

Proposition 2.23 is proved using *hierarchy paths*, as defined in [BHS19], which generalise the mapping class group notion from [MM00]. We will also use hierarchy paths, so we recall the definition ([BHS19, Defn. 4.2]):

Definition 2.24 (Hierarchy path)**.** Given $D \geqslant 0$, a *D–hierarchy path* in X is a (D, D)–quasi-isometric embedding $\gamma : [0, L] \to X$, for some $L \geqslant 0$, with the property that $\pi_W \circ \gamma$ is an unparametrised (D, D)–quasigeodesic in $\mathcal{C}W$ for all $W \in \mathfrak{S}$. ✽

Combining [BHS21, Lem. 1.37] with [BHS19, Thm. 4.4] yields:

Proposition 2.25. *There exists D, depending only on the HHS parameters, such that any two points in X are joined by a D–hierarchy path.*

Moreover, for any $D' \geqslant 0$, there exists $C \geqslant 0$, depending only on D' and the HHS parameters, such that a (D', D')–quasigeodesic $\gamma : [0, L] \to X$ is a D'-hierarchy path provided γ is C–quasimedian. Conversely, for all C, there exists D'' depending only on C and the HHS parameters such that any C–quasimedian (C, C)–quasiisometric embedding $\gamma : [0, L] \to X$ is a D''–hierarchy path.

Hierarchical quasiconvexity (equivalently, coarse median quasiconvexity) generalises quasiconvexity in hyperbolic spaces and "coarsifies" the notion of median convexity in median metric spaces. Accordingly, the coarse closest point projection to a quasiconvex subset of a hyperbolic space generalises to HQC subsets of HHSes in a way that mimics gate maps in median spaces, using the following construction from [BHS19, Sec. 5].

Construction 2.26 (HQC gate maps)**.** Let $(X, \mathfrak{S})$ be an HHS and let $Y \subseteq X$ be a κ–HQC subset. Define $\mathfrak{g}_Y : X \to 2^Y$ as follows. For each $U \in \mathfrak{S}$, let $p^Y_U : \mathcal{C}U \to 2^{\pi_U(Y)}$ be the coarse closest-point projection, which takes points to sets of diameter bounded in terms of E and $\kappa(0)$. Fix $x \in X$. Lemma 5.3 of [BHS19] provides a constant κ', depending on the HHS parameters and the function κ but not on x, such that any tuple $(q_U)_{U \in \mathfrak{S}}$ with $q_U \in p^Y_U(\pi_U(x))$ for all $U \in \mathfrak{S}$ is κ'–consistent. Hence there exists κ'', depending on the HHS parameters and κ only, such that there exists at least one $\bar{x} \in Y$ such that $d_U(\bar{x}, p^Y_U(\pi_U(x))) \leqslant \kappa''$ for all U. Let $\mathfrak{g}_Y(x)$ be the set of all such $\bar{x}$. As noted in [BHS19, Defn. 5.4, Lem. 5.5], up to uniformly enlarging κ'', we have $\mathsf{diam}(\mathfrak{g}_Y(x)) \leqslant \kappa''$ for all $x \in X$, and the coarse map $\mathfrak{g}_Y$ is (κ'', κ'')–coarsely lipschitz, κ''-quasimedian, and $d_Y(\mathfrak{g}_Y(y), y) \leqslant \kappa''$ for all $y \in Y$.

One can uniformly perturb a gate map $\mathfrak{g}_Y$ so that it is a coarsely lipschitz retraction onto Y. Thus HQC subsets are *coarse lipschitz retracts*.

Lemma 2.27 (Equivariant gate maps)**.** *Let $(X, \mathfrak{S})$ be an HHS. Let G be a group equipped with an action on $(X, \mathfrak{S})$ by HHS automorphisms. Suppose that $Y \subseteq X$ is κ–HQC and G–invariant. Then the gate map $\mathfrak{g}_Y : X \to 2^Y$ from Construction 2.26 is G–equivariant. Moreover, if the G–action on X is free, then there is a G–equivariant map $\mathfrak{g}'_Y : X \to Y$ such that $\sup_{x \in X} \mathsf{diam}(\mathfrak{g}_Y(x) \cup \mathfrak{g}'_Y(x))$ is bounded above in terms of κ and the HHS parameters.*

By Lemma 2.27, in any situation where the G–action on X is free and preserves Y, we will assume that $\mathfrak{g}_Y$ is a G–equivariant map $X \to Y$ (not just a coarse map) with the properties from Construction 2.26, with κ'' depending on the HHS parameters and κ only.

Proof of Lemma 2.27. From Definition 2.10 and the assumption that $GY = Y$, for all $U \in \mathfrak{S}$ and $g \in G$, we have $\pi_{gU}(Y) = g(\pi_U(Y))$. From the definition of coarse closest-point projection in a hyperbolic space, it follows that $p^Y_{gU}(\pi_{gU}(gx)) = p^Y_{gU}(g(\pi_U(x)) = g(p^Y_U(x))$ for all $x \in X$, so Construction 2.26 implies that $\mathfrak{g}_Y(gx) = g\mathfrak{g}_Y(x)$ for all $g \in G, x \in X$. Now suppose that the G–action on X is free. Let $\{x_i\}_{i \in I} \subseteq X$ contain exactly one point of X in each G–orbit, and for each $i \in I$, choose $\mathfrak{g}'_Y(x_i) \in \mathfrak{g}_Y(x)$ arbitrarily. Given $x \in X$, there is a unique $g \in G$ and $i \in I$ with $x = gx_i$, so set $\mathfrak{g}'_Y(x) = g\mathfrak{g}'_Y(x_i)$. This gives an equivariant map $\mathfrak{g}'_Y : X \to Y$, and $\mathsf{diam}(\mathfrak{g}_Y(x) \cup \mathfrak{g}'_Y(x)) = \mathsf{diam}(\mathfrak{g}_Y(x)) \leqslant \kappa''$ for all $x \in X$, as claimed. ☐

An important feature of hierarchically quasiconvex subsets is that they can be equipped with an induced HHS structure. The following holds by [BHS19, Lem. 5.6, Rem. 5.7]. The fact that the action of G on Y is by HHS automorphisms follows from the same considerations as in the equivariance part of the proof of Lemma 2.27.

Lemma 2.28 (HHS structure inherited by HQC set)**.** *Let $(X, \mathfrak{S})$ be an HHS, let $\kappa : [0, \infty) \to [0, \infty)$ be a function, and let $Y \subseteq X$ be a κ-hierarchically quasiconvex subspace. Let G be a group acting on X by HHS automorphisms such that $GY = Y$. Then there is an HHS structure $\mathfrak{S}_Y$ on Y such that the action of G on Y is by HHS automorphisms and the π_U–projection maps are coarsely surjective. The HHS parameters of $(Y, \mathfrak{S}_Y)$ depend only on the HHS parameters of $(X, \mathfrak{S})$ and κ.*

Remark 2.29. The structure $(Y, \mathfrak{S}_Y)$ from Lemma 2.28 can be constructed in various ways. If we do not require coarse surjectivity of the projections, then it has the following particularly simple description. Let $\mathfrak{S}_Y = \mathfrak{S}$ (as a set with relations $\sqsubseteq, \perp, \pitchfork$), for each $U \in \mathfrak{S}_Y$, let $\mathcal{C}_Y U = \mathcal{C}U$, and let $\pi^Y_U : Y \to \mathcal{C}U$ be $\pi^Y_U = \pi_U|_Y$. Recalling that the definition of an HHS does not require coarse surjectivity of projections, but only quasiconvexity of their images, $(Y, \mathfrak{S}_Y)$ is an HHS and in particular the distance formula (Theorem A.1) applies. This description is convenient for some distance estimates in the proof of Proposition 5.6, and we do not need it elsewhere.

However, if one wishes to stay within the subclass of HHSes with uniformly coarsely surjective projections, then one can simply normalise, as in [BHS19, Rem. 1.4]. In this procedure, if $V \sqsubsetneq U$ or $V \pitchfork U$, then replace ρ^V_U with its image under the coarse closest-point projection $\mathcal{C}U \to 2^{\pi_U(Y)}$ (using that $\pi_U(Y)$ is uniformly quasiconvex in $\mathcal{C}U$). ✽

2.7. **Existential hierarchical quasiconvexity.** Here is the weaker version of hierarchical quasiconvexity used in our main result:

Definition 2.30 (EHQC)**.** Let $k \geqslant 1, r \geqslant 0$. A subspace $Y \subseteq X$ is (k, r)*–existentially hierarchically quasiconvex (EHQC)* if, for all $y_1, y_2 \in Y$, there exists a k-hierarchy path γ from y_1 to y_2 with $\gamma \subseteq \mathcal{N}_r(Y)$. ✽

Remark 2.31. Observe that for any $k \geqslant 1, r \geqslant 0$, there exists k' such that Y is (k,r)–EHQC if and only if it is $(k',0)$–EHQC, since one can perturb a hierarchy path to lie in Y at the expense of a controlled change in the hierarchy path constant. ✽

The following is straightforward to check using Proposition 2.25 and Proposition 2.23; it motivates Definition 2.30, but we will not use it. Compare [HP22, Lem. 2.11, Prop. 2.8].

Corollary 2.32. *For all C, there exist k, r, depending on C and the HHS parameters, such that any C–coarsely connected C–coarse median subalgebra of X is (k,r)–EHQC.*

The converse to the preceding corollary is not true (consider codimension–1 affine subspaces of $\mathbb{R}^3$, and give $\mathbb{R}^3$ the ℓ_1 metric), so we have to work with the more general notion of EHQC, and the language of paths, instead of coarse medians and coarse subalgebras.

2.8. **Hulls in X and ∂X.** We recall *hierarchically quasiconvex hulls* from [BHS19, Sec. 6]:

Definition 2.33 (Hulls of interior points)**.** Let $A \subseteq X$. For all $U \in \mathfrak{S}$, let $\mathrm{hull}_U(A)$ be the union of all $(1,20E)$-quasi-geodesics in $\mathcal{C}U$ with endpoints in $\pi_U(A)$. Observe that $\mathrm{hull}_U(A)$ is a $10(E + \mathsf{Morse}_E(1,20E))$–quasiconvex subset of $\mathcal{C}U$. For any $M \geqslant 0$, let $H_M(A)$ be the set of all $x \in X$ such that $d_U(\pi_U(x), \mathrm{hull}_U(A)) \leqslant M$ for all $U \in \mathfrak{S}$. ✽

Hulls can be defined for sets that include points in the HHS boundary in a number of ways (see e.g. [Dur23, DZ22]). We only need hulls of pairs of points in $X \cup \partial X$:

Definition 2.34 (Hulls of pairs)**.** Fix $z_0 \in X$ and let $x, y \in X \cup \partial X$. Define the *hull* $\mathrm{hull}_{U,z_0}(x,y)$ of $\{x,y\}$ as follows, for each $U \in \mathfrak{S}$:

- If $x, y \in X$ then $\mathrm{hull}_{U,z_0}(x,y) := \mathrm{hull}_U(\{x,y\})$.
- If $x \in X$ and $y \in \partial X$ then $\mathrm{hull}_{U,z_0}(\{x,y\}) = \mathrm{hull}_U(x,y)$ is the union of all $(1,20E)$-quasi-geodesics from $\pi_U(x)$ to $\pi_{U,x}(y)$.
- If $x, y \in \partial X$ then $\mathrm{hull}_{U,z_0}(\{x,y\})$ is the union of all $(1,20E)$-quasi-geodesics from $\pi_{U,z_0}(x)$ to $\pi_{U,z_0}(y)$.

Given $M \geqslant 0$, let $H_{M,z_0}(x,y)$ be the set of points $p \in X$ such that $\pi_U(p)$ is in the M-neighbourhood of $\mathrm{hull}_{U,z_0}(x,y)$ for all $U \in \mathfrak{S}$. ✽

Remark 2.35. If $x, y \in X$ then $H_{M,z_0}(x,y) = H_M(\{x,y\})$. ✽

Lemma 6.2 in [BHS19] provides a constant θ' such that, for all $M \geqslant \theta'$ there is a function h_M such that $H_M(A)$ is h_M–hierarchically quasiconvex for any $A \subseteq X$, and, as noted in [CRHK24, Sec. 15], examining the proof of [BHS19, Lem. 6.2] shows that θ' and h_M depend only on the HHS parameters (using the fact that the number r_0 from the realisation theorem depends only on the HHS parameters). The proof of the same lemma also reveals:

Lemma 2.36. *There exists a constant $\theta \geqslant 0$, depending only on the HHS parameters, such that the following holds. For all $M \geqslant \theta$ and $z_0 \in X$, there exists a function $h_M : [0,\infty) \to [0,\infty)$, depending only on M and the HHS parameters, such that, if $H_{M,z_0}(x,y) \neq \emptyset$ then it is h_M-hierarchically quasiconvex.*

2.9. **Standard product regions.** Recall from [CRHK24, Sec. 17] and [BHS19, Sec. 5] the *standard product regions* in X: for each $U \in \mathfrak{S}$, the product region P_U is the set of $x \in X$ such that $d_V(x, \rho^U_V) \leqslant E$ for all $V \in \mathfrak{S}$ for which $U \subsetneq V$ or $U \pitchfork V$. By the definition of an HHS automorphism, we have $P_{gU} = gP_U$ for all $g \in G$, whenever G is a group acting on $(X, \mathfrak{S})$ by HHS automorphisms.

Next, recall the coarse product structure of P_U. Fix $q \in P_U$, and for each $V \in \mathfrak{S}$ let $q_V \in \pi_V(p)$. Then $F_U \times \{q\}$ is the following subset of P_U. Consider all $(p_V)_{V \sqsubseteq U} \in \prod_{V \sqsubseteq U} \mathcal{C}U$ that satisfy all of the E–consistency inequalities for pairs of nested or transverse pairs $V, V' \sqsubseteq$

U. For each such $(p_V)_{V \sqsubseteq U}$, extend to a tuple in $\prod_{V \in \mathfrak{S}} \mathcal{C}V$ by setting $p_V = q_V$ for $V \perp U$, and $p_V \in \rho^U_V$ for $U \subsetneq V$ or $U \pitchfork V$. Then $(p_V)_{V \in \mathfrak{S}}$ is E–consistent, and the realisation theorem provides a point $p \in P_U$ such that $d_V(p, p_V) \leqslant r_0 E$ for all $V \in \mathfrak{S}$. By the distance formula (see Theorem A.1 below), p is coarsely unique (constants depending only on the HHS parameters) but not necessarily unique. Let $F_U \times \{q\}$ be the set of all such p, as the tuple $(p_V)_{V \sqsubseteq U}$ varies. When q is not important, we will denote $F_U \times \{q\}$ by F_U and refer to it as a *coarse parallel copy of* F_U *in* P_U.

It is shown in, for instance, [CRHK24, Sec. 17], that the subspaces P_U and $F_U \times \{q\}$ and $\{p\} \times E_U$ are hierarchically quasiconvex, with hierarchical quasiconvexity functions depending only on the HHS parameters. In particular, there is no loss of generality in assuming that each of these subspaces is E–quasimedian quasiconvex, and we will do this to save notation.

2.10. **Canonical HHS cone-off.** The procedure for coning off certain hierarchically quasiconvex subsets of $(X, \mathfrak{S})$ to obtain new HHS structures was introduced in [BHS19, Section 2] and reproved in a slightly stronger form in [CRHK24, Section 19]. Here we adapt the cone-off construction to suit our purposes and use the statement from [CRHK24] to show that the adapted construction has the properties we need.

Definition 2.37 (Cone-off data)**.** Let $(X, \mathfrak{S})$ be an HHS, let d be the metric on X, let $A \to \mathrm{Isom}(X)$ be a group action by HHS automorphisms on $(X, \mathfrak{S})$, and let $\mathfrak{U} \subseteq \mathfrak{S}$ be an A–invariant subset that is downward-closed in the partial order $\sqsubseteq$. The tuple $((X, \mathfrak{S}), A, \mathfrak{U})$ is called *cone-off data.* ✽

In both cone-off constructions, one creates a new HHS, with an A–action by HHS automorphisms, where the hierarchically quasiconvex subsets $F_U \subset X$ for $U \in \mathfrak{U}$ have become bounded, and the new HHS index set is $\mathfrak{S} - \mathfrak{U}$. The new HHS, as a set, coincides with X, and the cone-off construction amounts to replacing d with a new metric. The difference between the construction from [BHS17b, CRHK24] and our adapted version is a technical difference in how the new metric is defined. With the original definition, as noted in [CRHK24], the group A does act on X, but the action is by uniform quasi-isometries, not isometries. The purpose of the adapted construction is to keep the A–action isometric.

We first recall the cone-off metric from [BHS17b, CRHK24]:

Definition 2.38 (Non-canonical $\mathfrak{U}$–cone-off)**.** Given $x, y \in X$, let $D(x, y) := d(x, y)$ unless there exists $U \in \mathfrak{U}$ and $e \in E_U$ such that $\{x, y\} \subset F_U \times \{e\}$, in which case $D(x, y) := \min\{1, d(x, y)\}$. The *non-canonical* $\mathfrak{U}$*–cone-off* is the metric space $\widehat{X} = (X, \widehat{d})$, where $\widehat{d}$ is the length metric on X induced by D. ✽

Remark 2.39 (Non-canonical)**.** The phrase *non-canonical* refers to the fact that the subspaces F_U of X are generally only coarsely well-defined. Because of the arbitrary choices involved in constructing the subspaces $F_U \times \{q\}$, the condition on pairs x, y ensuring that $D(x, y) \leqslant 1$ is not always A–invariant, so the A–action on $\widehat{X}$ may not be isometric. ✽

Definition 2.40 (Canonical $\mathfrak{U}$–cone-off)**.** Given $x, y \in X$, let $\check{D}(x, y) := d(x, y)$ unless both of the following hold:

- There exists $U \in \mathfrak{U}$ such that $x, y \in P_U$, and
- $d_V(x, y) \leqslant 10E$ for all $V \in \mathfrak{S}$ such that $V \perp U$.

If x, y satisfy both conditions, let $\check{D}(x, y) := \min\{1, d(x, y)\}$. Now let $\check{d}(x, y)$ be the length metric on X induced by $\check{D}$, and let $\check{X} = (X, \check{d})$, which is called *canonical* $\mathfrak{U}$*–cone-off.* ✽

Lemma 2.41. *The A–action on X induces an action of A on $\check{X}$ by isometries.*

Proof. This follows from the fact that the A–action on $(X, \mathfrak{S})$ is by HHS automorphisms and by isometries on (X, d). In particular, $gP_U = P_{gU}$, and $d_{gV}(gx, gy) = d_V(x, y)$, so $\check{D}$ is A–invariant, and hence $\check{d}$ is also. □

Lemma 2.42. *There exists C, depending only on the HHS parameters of $(X, \mathfrak{S})$, such that the set-theoretic identity map $\check{X} \to \widehat{X}$ is a (C, C)–quasi-isometry.*

Proof. Fix $x, y \in X$ and suppose that $D(x, y) \leqslant 1$. If $d(x, y) \leqslant 1$, then $\check{D}(x, y) \leqslant 1$. The other possibility is that there exists $U \in \mathfrak{U}$ and $q \in P_U$ such that $x, y \in F_U \times \{q\}$. By definition, this means that $x, y \in P_U$, and for all V such that $V \perp U$, we have $d_V(x, q_V) \leqslant E$ and $d_V(y, q_V) \leqslant E$, so $d_V(x, y) \leqslant 2E$, and hence $\check{D}(x, y) \leqslant 1$.

Conversely, suppose that $\check{D}(x, y) \leqslant 1$. If $d(x, y) \leqslant 1$, then $D(x, y) \leqslant 1$. Otherwise, there exists $U \in \mathfrak{U}$ such that $x, y \in P_U$ and $d_V(x, y) \leqslant 10E$ whenever $V \perp U$. Consider the subspace $F_U \times \{x\}$. As noted just after Lemma 17.1 in [CRHK24] (or see [BHS19, Sec. 5]), $F_U \times \{x\}$ is uniformly hierarchically quasiconvex in $(X, \mathfrak{S})$, and admits a gate map $\mathfrak{g} : X \to F_U \times \{x\}$ that is uniformly coarsely lipschitz and has the following properties, for all $z \in X$:

- If $V \in \mathfrak{S}$ and $V \sqsubseteq U$, then $d_V(z, \mathfrak{g}(z)) \leqslant r_0 E$.
- If $V \in \mathfrak{S}$ and $V \perp U$, then $d_V(\mathfrak{g}(z), x) \leqslant r_0 E$.
- If $V \in \mathfrak{S}$ and $U \sqsubsetneq V$ or $U \pitchfork V$, then $d_V(\mathfrak{g}(z), \rho^U_V) \leqslant r_0 E$.

In particular, $d_V(x, \mathfrak{g}(x)) \leqslant r_0 E$ for all $V \in \mathfrak{S}$, so $d(x, \mathfrak{g}(x)) \leqslant \theta_u(r_0 E)$. Similarly, $d_V(y, \mathfrak{g}(y)) \leqslant 10E + r_0 E$ for all V, so $d(y, \mathfrak{g}(y)) \leqslant \theta_u(10E + r_0 E)$. Now, $D(\mathfrak{g}(x), \mathfrak{g}(y)) \leqslant 1$ by the definition of D, so $D(x, y) \leqslant \theta_u(r_0 E) + \theta_u(10E + r_0 E) + 1$, which depends only on the HHS parameters.

We have shown that the identity map $(X, \widehat{d}) \to (X, \check{d})$ and its inverse are both uniformly coarsely lipschitz, so they are quasi-isometries, as required. □

Corollary 2.43. *Let $(X, \mathfrak{S})$, A, and $\mathfrak{U}$ be as in Definition 2.37. Then there is an HHS $(\check{X}, \mathfrak{S} - \mathfrak{U})$, with HHS parameters depending only on the HHS parameters of $(X, \mathfrak{S})$, such that A acts on $(\check{X}, \mathfrak{S} - \mathfrak{U})$ by HHS automorphisms and on $\check{X}$ is by isometries, and there is an A–equivariant quasi-isometry $\check{X} \to \widehat{X}$ with constants depending only on the HHS parameters.*

Finally, the identity map $c : X \to \check{X}$ is coarsely lipschitz and quasimedian, with the constants for both properties depending only on the HHS parameters of $(X, \mathfrak{S})$.

Proof. By Proposition A.4, we have a constant C, depending only on the HHS parameters of $(X, \mathfrak{S})$, such that A acts by (C, C)–quasi-isometries on the non-canonical cone-off $\widehat{X}$, and this action is by HHS automorphisms on $(\widehat{X}, \mathfrak{S} - \mathfrak{U})$.

Lemma 2.42 implies that the canonical cone-off $(\check{X}, \mathfrak{S} - \mathfrak{U})$ is an HHS with uniform HHS parameters, defining projections $\pi'_W : \check{X} \to \mathcal{C}W$, $W \in \mathfrak{S} - \mathfrak{U}$ by composing the projections $\pi_W : \widehat{X} \to \mathcal{C}W$ with the identity $\check{X} \to \widehat{X}$ (see also [BHS19, Prop. 1.10]). The A–action on $\check{X}$ is isometric by Lemma 2.41. The action is by HHS automorphisms since that holds for $(\widehat{X}, \mathfrak{S} - \mathfrak{U})$, and passing to $\check{X}$ did not change the index set or the hyperbolic spaces, and did not change the projection maps at the set-theoretic level.

Finally, the map identity map c is coarsely lipschitz with uniform constants as an immediate consequence of Definition 2.40 and the fact that X is a uniform quasigeodesic space. The fact that c is quasimedian follows from the characterisation of the coarse median in an HHS (Construction 2.17) and the fact that, for $U \in \mathfrak{S} - \mathfrak{U}$, the projections to $\mathcal{C}U$ coincide for the HHS structures on X and $\check{X}$. This completes the proof. □

Remark 2.44. In our applications, the input HHS X will be the vertex set of a graph, and d will be the usual graph metric obtained by assigning edges length 1. In this setting, $\check{X}$

is obtained by adding unit length edges joining any two vertices x, y with $\breve{D}(x,y) \leqslant 1$, and taking the resulting graph metric on the vertex set. ✽

2.11. **Coarse injectivity and consequences.** Fix an HHS $(X, \mathfrak{S})$ and let σ be the metric on X given by Theorem A.5, which is M_1–coarsely injective; coarse injectivity originates in [CE07] and we refer to [HHP23, Sec. 1.1] for an explicit definition, which we do not need here. The same theorem says that σ makes the identity $(X,d) \to (X,\sigma)$ an M_2–quasi-isometry, where M_1, M_2 depend only on the HHS parameters, and that any subgroup of $\mathrm{Isom}(X,d)$ acting on $(X,\mathfrak{S})$ by HHS automorphisms also acts by isometries on (X,σ).

Let $(\mathrm{Env}(X), d_\infty)$ be the injective hull of (X,σ) and recall that the canonical isometric embedding $i : (X,\sigma) \hookrightarrow (\mathrm{Env}(X), d_\infty)$ is $\mathrm{Isom}(X,\sigma)$-equivariant (see [Lan13]). Hence G acts by isometries on $\mathrm{Env}(X)$ and i is G–equivariant. The next corollary follows from the proof of Proposition 3.12 in [CCG+25].

Corollary 2.45. *The image $i(X)$ is M_1-coarsely dense in* $\mathrm{Env}(X)$.

The main consequence of coarse injectivity that we shall apply later is:

Corollary 2.46 (Uniformly bounded orbits)**.** *Let $(X,\mathfrak{S})$ be an HHS and assume X is a graph. There exists a constant B depending only on the HHS parameters such that, if $H \leqslant \mathrm{Isom}(X,d)$ is a group of HHS automorphisms of $(X,\mathfrak{S})$ acting on X with bounded orbits, then there exists $x_0 \in X$ such that* $\mathsf{diam}(H \cdot x_0) \leqslant B$.

Proof. By [Lan13, Proposition 1.2], there is a point $f \in \mathrm{Env}(X)$ which is fixed by G. Let $x \in X$ be such that $d_\infty(i(x), f) \leqslant M_1$. Then

$$\mathsf{diam}_d(G \cdot x) \leqslant M_2\, \mathsf{diam}_\sigma(G \cdot x) + M_2 \leqslant 2M_2M_1 + M_2 =: B,$$

which depends only on the HHS parameters since the same is true of M_1, M_2. □

2.12. **Simplifying actions.** Here are two more or less standard lemmas that will be convenient in places where we want to perturb maps to make them equivariant.

Lemma 2.47. *For every $q \geqslant 1$, there exists $r \geqslant 1$, depending only on q, such that the following holds. Let (X,d) be a q–quasigeodesic space on which the group G acts by isometries. Then there is a graph Ω such that G acts on Ω by graph automorphisms and there is a G–equivariant map $f : X \to \Omega$ that is an r–coarsely surjective (r,r)–quasi-isometry, where Ω is equipped with the usual path-metric in which all edges have unit length. Moreover, Ω has vertex set X and we may take f to be the inclusion $X \to \Omega$.*

Proof. This follows from [CdlH16, Lem. 3.B.6]. □

Remark 2.48. Let $(X,\mathfrak{S})$ be an HHS on which G acts isometrically by HHS automorphisms. Then Lemma 2.47 allows us to remetrise X as the vertex set of a graph, without changing the projections $\pi_U : X \to \mathcal{C}U$; by [BHS19, Prop. 1.10], $(X,\mathfrak{S})$ is still an HHS, it is immediate from Definition 2.10 that the action is still by isometric HHS automorphisms, and the new HHS parameters just depend on the old ones, via the quasi-isometry constant r from Lemma 2.47, which just depends on the old HHS parameter q from Subsection 2.2. In a few places, it will be convenient to assume that X is a graph, and this observation enables it. ✽

Lemma 2.49 (Freeing actions)**.** *Let Ω be a simplicial graph and let G act by graph automorphisms on Ω without inverting edges. Then there is a graph $\widehat{\Omega}$, a free G–action $G \to \mathrm{Aut}(\widehat{\Omega})$, and a G–equivariant surjective map $f : \widehat{\Omega} \to \Omega$ that is a $(1,10)$–quasi-isometry taking vertices to vertices and edges to edges or vertices.*

Proof. Let V be the vertex set of Ω and let E be the set of edges. Let $\Delta(G) = \mathrm{Cay}(G, G)$ be a complete graph with a free, vertex-transitive G–action. The G–action on G by left multiplication extends to an action on $\Delta(G)$, and we are given a G–action on Ω. These two actions yield a diagonal action on the complex $\Delta(G) \times \Omega$ by combinatorial automorphisms.

Let $\{v_i\}_{i\in I} \subseteq V$ contain exactly one element of V in each G–orbit. Define $\widehat{V} \subseteq G \times V$ as follows. Given $g \in G$ and $v \in V$, let $i \in I$ be the unique element such that $v = hv_i$ for some $h \in G$. Then $(g, v) \in \widehat{V}$ if and only if $h^{-1}g \in \mathrm{Stab}_G(v_i)$. Note that if $(g, hv_i) \in \widehat{V}$ and $a \in G$, then $(ag, ahv_i) \in \widehat{V}$, so the G–action preserves $\widehat{V}$. The set $\widehat{V}$, with this G–action, will be the vertex set of $\widehat{\Omega}$. The G–action on $\widehat{V}$ is free because the action on the first coordinate is free.

Declare $(g, v), (h, w) \in \widehat{V}$ to be adjacent in $\widehat{\Omega}$ if and only if $v = w$ or $\{v, w\} \in E$. The G–action on $\widehat{V}$ preserves these relations and thus extends to a free G–action on $\widehat{\Omega}$.

The natural projection $\Delta(G) \times \Omega \to \Omega$ restricts to a G–equivariant map $f : \widehat{\Omega} \to \Omega$ taking vertices to vertices. Each edge of $\widehat{\Omega}$ joining vertices with the same second coordinate gets collapsed to a vertex, and the remaining edges are sent isometrically to edges. Hence f is 1–lipschitz. On the other hand, the map $\bar{f} : V \to \widehat{V}$ given by $\bar{f}(hv_i) = (h, hv_i)$ (the map is determined by choices of coset representatives for the $\mathrm{Stab}_G(v_i)$) defines a section of f that is 1–lipschitz and satisfies $d_{\widehat{\Omega}}(x, \bar{f}f(x)) \leqslant 1$ for all x. This completes the proof. □

2.13. **Virtually abelian and crystallographic groups.** We recall some standard facts about virtually abelian and crystallographic groups. Fix $n \geqslant 0$ and let A be a virtually $\mathbb{Z}^n$ group, so that there is a short exact sequence $A_0 \hookrightarrow A \twoheadrightarrow F$, where F is finite and $A_0 \cong \mathbb{Z}^n$.

Definition 2.50 (Point group of A)**.** The conjugation action of A on A_0 gives a homomorphism $A \to GL_n(\mathbb{Z})$ whose image we denote by $\bar{F}$ and call the *point group* of A. ✽

Note that $A \to GL_n(\mathbb{Z})$ factors through the quotient $A \to F$, so $\bar{F}$ is finite. Moreover, if $A_0' \leqslant A_0$ is another finite-index free abelian subgroup that is normal in A, then the conjugation actions of A on A_0 and A_0' make the map $\iota : A_0' \hookrightarrow A_0$ an A–equivariant map, where A acts on A_0' via the point group $\bar{F}' = \mathrm{im}(A \to Aut(A_0'))$ and on A_0 via $\bar{F}$. But ι extends to an isomorphism $\mathbb{R}^n \to \mathbb{R}^n$, which therefore conjugates $\bar{F}'$ to $\bar{F}$. This implies that the point group $\bar{F} \leqslant GL_n(\mathbb{R})$ is, up to conjugacy, independent of the choice of A_0.

Virtually $\mathbb{Z}^n$ groups are related to n–dimensional crystallographic groups by the following classical fact [Zas48]:

Lemma 2.51. *If A is a virtually $\mathbb{Z}^n$ group, then there is a finite normal subgroup $K \leqslant A$ and a monomorphism $\phi : A/K \to \mathrm{Isom}(\mathbb{E}^n)$ whose image is n–dimensional crystallographic, i.e. $\phi(A/K)$ acts faithfully on $\mathbb{E}^n$ and this action is proper and cocompact.*

In Section 7, we will use these facts via the following lemma:

Lemma 2.52. *Let $n \geqslant 0$ and let K be a finite group. Then there are finitely many abstract isomorphism classes of groups A such that $K \lhd A$ and A/K is isomorphic to an n–dimensional crystallographic group.*

Proof. Fix n. Let A be as in the statement and let $C = A/K$. By Bieberbach's theorems, C belongs to one of finitely many isomorphism classes. Since K is finite and A is finitely generated, there are finitely many homomorphisms $C \to \mathrm{Out}(K)$. The centre $Z(K)$ of K is finite, so for each way of regarding $Z(K)$ as a C–module, $H^2(C, Z(K))$ is finite, and hence yields at most finitely many possibilities for A. □

In [PS23], Petyt and Spriano characterised crystallographic HHGs, using the following notion, which was also used in [Hag14] and [Hod23], respectively, to characterise crystallographic groups that are cocompactly cubulated and act geometrically on injective spaces.

Definition 2.53 (Hyperoctahedral group)**.** Let A be a virtually $\mathbb{Z}^n$ group and let $\bar{F}$ be the point group of A. Then A is *hyperoctahedral* if $\bar{F}$ is conjugate in $GL_n(\mathbb{R})$ into $O_n(\mathbb{Z})$. ✽

We will use this notion in Section 7, when we study crystallographic *subgroups* of HHGs.

3. An equivariant median model

Our aim in this section is to prove the following equivariant version of the cubical approximation theorem. By [BHS19, Rem. 1.4], in this section, we can and shall assume that the maps $\pi_U : X \to \mathcal{C}U$, $U \in \mathfrak{S}$ are E–coarsely surjective.

Theorem 3.1 (Equivariant median model)**.** *Let $(X, \mathfrak{S})$ be an HHS and suppose there exists $M \geqslant 0$ and $p^-, p^+ \in X \cup \partial X$ such that $X = H_{M,z_0}(p^-, p^+)$ for any $z_0 \in X$. Let G be a group acting on X by HHS automorphisms and, if $p^-, p^+ \in \partial X$, suppose G is virtually abelian. Then there is a complete, connected, finite rank, proper median space Q, equipped with an action of G, and a G-equivariant L'-quasimedian (L', L')-quasi-isometry $\Psi : X \to Q$, where the constant $L' \geqslant 1$ depends only on the HHS parameters and M.*

To support the proof of Theorem 6.1.(4) we also prove the following in Section 3.9.

Proposition 3.2. *Let $(X, \mathfrak{S}), M, p^\pm$, and G be as in Theorem 3.1. Suppose that $p^\pm \in \partial X$, and $\operatorname{supp}(p^-) = \operatorname{supp}(p^+)$, and $\pi_U(G \cdot z_0)$ is unbounded for all $U \in \operatorname{supp}(p^\pm)$ and some (equivalently any) $z_0 \in X$. Let Q be the G–median space provided by Theorem 3.1. Then there exist median spaces $Q_1, \ldots, Q_r$ such that:*

- *each Q_i is quasi-isometric to $\mathbb{R}$, and*
- *G acts by isometries on $\prod_{i=1}^r Q_i$, preserving the product structure, and there is a G–equivariant isometry $Q \to \prod_{i=1}^r Q_i$,*
- *if $G' \leqslant G$ is the finite index subgroup which acts trivially on the set of factors, then the G'-action on each Q_i is cocompact.*

Moreover, there exists $B < \infty$ such that $\mathsf{diam}(\mathcal{C}V) < B$ for all $V \in \mathfrak{S} - \operatorname{supp}(p^\pm)$.

Remark 3.3. The quasi-isometry constants and the constant B in Proposition 3.2 are not uniform; they depend on more than the HHS constants and M. ✽

3.1. **Quasi-linear domains.** We need to fix quasi-isometries from the domains of $\mathfrak{S}$ to lines, rays and intervals in an equivariant way. We start with some lemmas.

Lemma 3.4. *Let $(X, \mathfrak{S})$ be an HHS and let $p^\pm, z_0, G$ and M be as in Theorem 3.1. If $p^+ \in \partial X$ (resp. $p^- \in \partial X$) and $U \in \operatorname{supp}^\perp(p^+)$ (resp. $U \in \operatorname{supp}^\perp(p^-)$) then $\mathsf{diam}(\mathcal{C}U) \leqslant \ddot{M}$, where $\ddot{M} \geqslant M$ depends on M and the HHS parameters only.*

Proof. Since $\pi_U(X) = \pi_U(H_{M,z_0}(p^-, p^+))$ for all $z_0 \in X$ and π_U is E–coarsely surjective, $\mathcal{C}U$ is $(M + E)$–Hausdorff close to $\operatorname{hull}_{U,z_0}(p^-, p^+)$ for all $U \in \mathfrak{S}$, by Definition 2.34.

Suppose $p^+ \in \partial X$ and $U \in \operatorname{supp}^\perp(p^+)$. Fix $z_0 \in X$. By Definition 2.3, $\pi_{U,z_0}(p^+) = \pi_U(z_0)$. By Definition 2.34, $\operatorname{hull}_{U,z_0}(p^-, p^+)$ is therefore the union of all $(1, 20E)$–quasigeodesics from $\pi_U(z_0)$ to $q := \pi_{U,z_0}(p^-)$. If $p^- \in \partial X$ and $U \in \operatorname{supp}^\perp(p^-)$, then $q = \pi_U(z_0)$, so $\operatorname{hull}_{U,z_0}(p^-, p^+) \subseteq \mathcal{N}_{100E}(\pi_U(z_0))$, which has uniformly bounded diameter.

Otherwise, q is independent of the choice of z_0 (see Definition 2.3). In this case, since $X = H_{M,z_0}(p^-, p^+)$ for every choice of z_0, the diameter of $\operatorname{hull}_{U,z_0}(p^-, p^+)$, and hence $\mathcal{C}U$, is bounded in terms of M and E. □

Lemma 3.5. *For each $k_1 \geqslant 1$, there exists $k \geqslant 1$ such that the following holds. Let A be a group and let Σ be a k_1-quasi-line with an isometric A–action. Suppose that either the action has bounded orbits or A is virtually abelian. Then A acts isometrically on $\mathbb{R}$ and there is an A-equivariant k-quasi-isometry $f : \Sigma \to \mathbb{R}$.*

Proof. Fix $k_1 \geqslant 1$ and let A, Σ be as in the statement. Assume momentarily that A fixes the ends of Σ. By [Ker23, Cor. 4.12], there exists $k_2 \geqslant 1$, depending only on k_1, and a $(1, k_2)$-quasi-isometry $f_1 : \Sigma \to \mathbb{R}$. Without loss of generality, $0 = f_1(x_0)$ for some $x_0 \in \Sigma$. For all $a \in A$, define $\ell(a) := \lim_{n\in\mathbb{N}} \frac{1}{n} f_1(a^n x_0)$. Since f_1 is a rough isometry, $|\ell(a)| = \lim_{n\in\mathbb{N}} \frac{1}{n} d_\Sigma(x_0, a^n x_0) < \infty$ for all $a \in A$.

Claim 1. The map $\ell : A \to \mathbb{R}$ is a homomorphism.

Proof. Let $a, b \in A$. Then, for all $n \in \mathbb{N}$,

$$f_1(a^n x_0) + f_1(b^n x_0) - 3k_2 \leqslant f_1(a^n b^n x_0) \leqslant f_1(a^n x_0) + f_1(b^n x_0) + 3k_2,$$

so the claim follows from [Cal09, Prop. 2.65] in the virtually abelian case. Otherwise, our assumption that A has bounded orbits and fixes the endpoints implies that ℓ is the trivial map, which is a homomorphism.

■

There exists a constant k_3 depending only on k_1 such that, for any ball $B_\Sigma(x, k_3) \subseteq \Sigma$, the complement $\Sigma - B_\Sigma(x, k_3)$ has precisely two unbounded components.

Claim 2. For all $a \in A$ and $x \in \Sigma$ we have $f_1(ax) - f_1(x) - 2k_2 - k_3 \leqslant \ell(a) \leqslant f_1(ax) - f_1(x) + 2k_2$.

Proof. Firstly,

$$|\ell(a)| = \lim_{n\to\infty} \frac{d_\Sigma(x_0, a^n x_0)}{n} \leqslant \lim_{n\to\infty} \frac{2d_\Sigma(x_0, x) + d_\Sigma(x, a^n x)}{n} = \lim_{n\to\infty} \frac{d_\Sigma(x, a^n x)}{n}.$$

A symmetric argument shows that $\lim_{n\to\infty} \frac{1}{n} d_\Sigma(x, a^n x) \leqslant |\ell(a)|$. It follows that $\ell(a) = \lim_{n\to\infty} \frac{1}{n}(f_1(a^n x) - f_1(x))$. Now, using the triangle inequality and the fact that f_1 is a rough isometry,

$$f_1(a^n x) - f_1(x) \leqslant \sum_{i=0}^{n-1} f_1(a^{i+1} x) - f_1(a^i x) \leqslant n f_1(a x_0) + 2nk_2$$

for all $n \in \mathbb{N}$, which proves the second inequality.

To prove the first inequality, observe that, if $d_\Sigma(x, ax) > k_3$ then either $f_1(a^n x) < f_1(a^{n+1} x)$ for all $n \in \mathbb{Z}$ or $f_1(a^n x) > f_1(a^{n+1} x)$ for all $n \in \mathbb{Z}$. It follows that

$$f_1(a^n x) - f_1(x) = \sum_{i=0}^{n-1} f_1(a^{i+1} x) - f_1(a^i x) \geqslant n f(a x_0) - 2nk_2. \qquad ■$$

Let $\{x_i\}_{i\in I} \subseteq \Sigma$ contain exactly one point in each A–orbit. Define $f : \Sigma \to \mathbb{R}$ by $f(a x_i) = \ell(a) + f_1(x_i)$ for $i \in I$ and $a \in A$. Observe that if $a x_i = x_i$, then $a^n x_i = x_i$ for all n, and hence $f_1(a^n x_0)$ is bounded, so $\ell(a) = 0$. Thus, if $a, b \in A$ satisfy $a x_i = b x_i$, then $\ell(a) = \ell(b)$, and therefore f is a well-defined map. Claim 2 implies that f is a quasi-isometry with constants depending only on k_1. Finally, the map $A \times \mathbb{R} \ni (a, t) \mapsto \ell(a) + t$ defines an isometric action of A on $\mathbb{R}$, by Claim 1, and f is equivariant with respect to this action, by construction.

If A permutes the ends of Σ, let $A' \leqslant A$ be the index–2 subgroup fixing the ends and let $f : \Sigma \to \mathbb{R}$ and $\ell : A' \to \mathbb{R}$ be the homomorphism obtained by applying the preceding construction to A'. Then the standard induced representation construction gives the desired action $A \to \mathbb{R} \rtimes \mathbb{Z}/2$ making the map $\Sigma \to \mathbb{R}$ equivariant. □

Proposition 3.6. *Let G and $(X, \mathfrak{S})$ be as in Theorem 3.1. There exists a constant $L \geqslant 1$ depending only on the HHS parameters and M such that the following holds. There exists an interval, line or ray T_U for each $U \in \mathfrak{S}$ and an L–coarsely surjective (L, L)–quasi-isometry $f_U : \mathcal{C}U \to T_U$ such that:*

- *for all $g \in G$ and $U \in \mathfrak{S}$ there is an isometry $g : T_U \to T_{gU}$;*
- *for all $g, h \in G$ and $U \in \mathfrak{S}$ the isometry $gh : T_U \to T_{ghU}$ is the composition $g \circ h$.*
- *for all $U \in \mathfrak{S}$ the map f_U is $\mathrm{Stab}_G(U)$–equivariant.*
- *T_U is unbounded if and only if $U \in \mathrm{supp}(q)$ for some $q \in \{p^-, p^+\} \cap \partial X$.*

Moreover, letting $p^\pm \in X \cup \partial X$ be as in Theorem 3.1, G has a finite index subgroup fixing $\{p^-, p^+\} \cap \partial X$ pointwise.

Proof. In this proof, *uniform* quantities are those bounded in terms of the HHS parameters and M. Recall that $X = H_{M,z_0}(p^-, p^+)$ for all $z_0 \in X$, by hypothesis. For each $U \in \mathfrak{S}$, our standing assumption ensures that $\mathcal{C}U = \mathcal{N}_E(\pi_U(H_{M,z_0}(p^-, p^+)))$. By Definition 2.34, it follows that $\mathcal{C}U = \mathcal{N}_{E+M}(\mathrm{hull}_{U,z_0}(p^-, p^+))$.

We first deduce from this that each $\mathcal{C}U$ is uniformly quasi-isometric to an interval, line, or ray. By the preceding observation, it suffices to show that each $\mathrm{hull}_{U,z_0}(p^-, p^+)$ has this property. If $p^-, p^+ \in X$, then by Remark 2.35, $\mathcal{C}U$ is at uniformly bounded Hausdorff distance from some geodesic joining $\pi_U(p^-)$ to $\pi_U(p^+)$, as required.

Therefore, suppose that $p^+ \in \partial X$. If $p^+ \in \partial X$ but $p^- \in X$, then for all U, $\mathcal{C}U$ is, by the second point in Definition 2.34, at uniformly bounded Hausdorff distance from a geodesic interval or ray, as required. If $p^-, p^+ \in \partial X$, then the third point in Definition 2.34 yields the same conclusion. (If $U \in \mathrm{supp}(p^-) \cap \mathrm{supp}(p^+)$, we implicitly used that $\pi_U(p^-) \neq \pi_U(p^+)$, but this follows from Definition 2.34 since $X \neq \varnothing$ and $X = H_{M,z_0}(p^-, p^+)$.)

We now construct T_U and f_U for each $U \in \mathfrak{S}$. First, we have just shown that there is a uniform constant $k_1 \geqslant 1$ such that for all $U \in \mathfrak{S}$, $\mathcal{C}U$ is (k_1, k_1)–quasi-isometric to an interval, ray, or line. Second, we will use the following observation: if $p^-, p^+ \in X$, then, as noted above, $\mathcal{C}U$ is bounded for all U. Hence, if any $\mathcal{C}U$ is unbounded, then one of $p^\pm \in \partial X$ and therefore G is virtually abelian, by the hypothesis in Theorem 3.1.

Fix $U \in \mathfrak{S}$. We define T_U and f_U as follows.

If $\mathcal{C}U$ is two-ended, then Lemma 3.5 provides a homomorphism $\phi_U : \mathrm{Stab}_G(U) \to \mathrm{Isom}(\mathbb{R})$ and a quasi-isometry $f_U : \mathcal{C}U \to \mathbb{R} =: T_U$ such that $f_U(gc) = \phi_U(g) f_U(c)$ for all $g \in \mathrm{Stab}_G(U)$ and $c \in \mathbb{R}$.

If $\mathcal{C}U$ is either one-ended or bounded, then at least one of $\pi_U(p^\pm)$ is a uniformly bounded subset of $\mathcal{C}U$, and the set $\pi_U(p^-) \cup \pi_U(p^+)$ is $\mathrm{Stab}_G(U)$–invariant and each $g \in \mathrm{Stab}_G(U)$ either stabilises $\pi_U(p^\pm)$ or sends it bijectively to $\pi_U(p^\mp)$. Form a new space $\mathcal{C}'U$ as follows: if $\pi_U(p^+)$ (respectively, $\pi_U(p^-)$) is contained in X, then add a vertex v^+ (respectively v^-) joined by an edge to each element of $\pi_U(p^+)$ (respectively, $\pi_U(p^-)$), and then attach a copy of $[0, \infty)$ by identifying 0 with v^+; if v^- has been defined, also attach a differente copy of $[0, \infty)$ there. Then $\mathcal{C}'U$ is a uniform quasiline in which $\mathcal{C}U$ is isometrically embedded as a G–invariant subset, and the $\mathrm{Stab}_G(U)$ action extends naturally over $\mathcal{C}'U$. Apply Lemma 3.5 to get a $\mathrm{Stab}_G(U)$–action on $\mathbb{R}$ and an equivariant map $\mathcal{C}'U \to \mathbb{R}$, which we then restrict to $\mathcal{C}U$.

Applying the above construction to G–orbit representatives in $\mathfrak{S}$ and extending equivariantly in the usual way yields isometries $g : T_U \to T_{gU}$ with the equivariance properties required by the statement.

Finally, we characterise the $U \in \mathfrak{S}$ for which $\mathcal{C}U$ (equivalently T_U) is unbounded. If $p^\pm \in \partial X$ and $U \in \mathrm{supp}(p^\pm)$, then $\mathcal{C}U$ is unbounded since $\pi_{U,z_0}(p^\pm) \in \partial \mathcal{C}U$ in this case. Conversely, suppose that $\mathcal{C}U$ is unbounded. As noted above, this implies that we can assume $p^+ \in \partial X$. Lemma 3.4 implies that $U \notin \mathrm{supp}^\perp(p^\pm)$. Hence it suffices to consider the case where $U \notin \mathrm{supp}(p^\pm) \cup \mathrm{supp}^\perp(p^\pm)$, but in this case, Definition 2.3 and Definition 2.34 imply that $\mathcal{C}U$ is bounded.

Since G acts by HHS automorphisms, the set of $U \in \mathfrak{S}$ for which $\mathcal{C}U$ is unbounded is G–invariant, which implies that $\mathrm{supp}(p^-) \cup \mathrm{supp}(p^+)$ is G–invariant (here we take $\mathrm{supp}(p^\pm) =$

$\varnothing$ for $p^{\pm} \in X$). Since $\mathrm{supp}(p^{\pm})$ consists of pairwise-orthogonal elements, $|\mathrm{supp}(p^-) \cup \mathrm{supp}(p^+)|$ is finite (in fact bounded in terms of the complexity of $(X, \mathfrak{S})$), and hence $G' = \bigcap_{U \in \mathrm{supp}(p^-) \cup \mathrm{supp}(p^+)} \mathrm{Stab}_G(U)$ has finite (in fact, uniformly bounded) index in G. For each $U \in \mathrm{supp}(p^-) \cup \mathrm{supp}(p^+)$, the G'–action on $\mathcal{C}U$ extends, as usual, to an action on $\partial\mathcal{C}U$, which consists of at most two points. Hence passing to the intersections of the kernels of the G'–actions on these $\partial\mathcal{C}U$ yields the desired finite-index subgroup. $\square$

3.2. **The product of trees $\mathcal{Y}$.** Let L and $\{f_U : \mathcal{C}U \to T_U : U \in \mathfrak{S}\}$ be as in Proposition 3.6. Let $U \in \mathfrak{S}$. If $U \notin \mathrm{supp}^{\perp}(p^{\pm})$, then by Definition 2.3, $\pi_U(p^{\pm})$ is defined independently of z_0, and $\pi_{U,z_0}(p^{\pm}) = \pi_U(z_0)$ otherwise. Up to enlarging L by a uniformly bounded amount, we can (and do) assume that, if $\pi_U(p^{\pm}) \subseteq \mathcal{C}U$, then $f_U(\mathcal{N}_{10(EM+E)}(\pi_U(p^{\pm})))$ is an endpoint of T_U. Denote this point by $p_U^{\pm}$.

If $\pi_U(p^{\pm}) \in \partial\mathcal{C}U$ then let $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}U$ be a sequence which converges to $\pi_U(p_U^{\pm})$ and let $p_U^{\pm} \in \partial T_U$ be the limit of $(f_U(x_n))_{n\in\mathbb{N}}$ in ∂T_U, which is independent of the choice of $(x_n)_{n\in\mathbb{N}}$.

Enlarging L be a uniform amount again, we can fix, for each $U \in \mathfrak{S}$, a quasi-inverse $f_U^{-1} : T_U \to \mathcal{C}U$ for f_U and assume it is an (L, L)-quasi-isometry. In general it will not be possible to make f_U^{-1} $\mathrm{Stab}_G(U)$–equivariant.

Definition 3.7. Let C_0, C_1 be as in Section 2.5. Fix constants $E' := L(5E + L + 1)$ and $K := 100L(10(EM + E) + E) + 100E' + L^2 + L(C_0 + C_1) + \ddot{M}$. Let $\mathfrak{U} := \{U \in \mathfrak{S} : \mathsf{diam}(T_U) \geqslant K\}$. We say that $U \in \mathfrak{U}$ is $\sqsubseteq_{\mathfrak{U}}$-*minimal* if it is $\sqsubseteq$-minimal in $\mathfrak{U}$. ✽

Remark 3.8.

- The set $\mathfrak{U}$ is G–invariant, by Proposition 3.6.
- If $U \in \mathfrak{U}$ and $p^{\pm} \in \partial X$ then it follows from Lemma 3.4 that $U \notin \mathrm{supp}^{\perp}(p^{\pm})$. Therefore $\pi_U(p^{\pm})$ is well-defined (independently of a choice of basepoint) for any $p^{\pm} \in X \cup \partial X$ and $U \in \mathfrak{U}$. ✽

Let $\mathcal{Y} := \prod_{U \in \mathfrak{U}} T_U$ and define an action of G on $\mathcal{Y}$ as follows. Given $(x_U)_{U\in\mathfrak{U}} \in \mathcal{Y}$ and $g \in G$, let $y_U := g x_{g^{-1}U}$ for each $U \in \mathfrak{U}$ and let $g(x_U)_{U\in\mathfrak{U}} := (y_U)_{U\in\mathfrak{U}}$, where $g : T_{g^{-1}U} \to T_U$ is the isometry from Proposition 3.6.

Definition 3.9 (Interval projections)**.** Given distinct $U, V \in \mathfrak{S}$ that are not orthogonal, define the *interval projections* δ_V^U, δ_U^V as follows:

- If $U \pitchfork V$ or $U \subsetneq V$ then define $\delta_V^U := \mathrm{hull}(f_V(\rho_V^U))$.
- If $U \subsetneq V$ then define $\delta_U^V : T_V \to 2^{T_U}$ by $\delta_U^V := f_U \circ \rho_U^V \circ f_V^{-1}$. ✽

Lemma 3.10 (Consistent projections)**.** *For all $U, V, W \in \mathfrak{U}$ we have:*

(1) If $U \pitchfork V$ then either $\delta_V^U = \{p_V^-\}$ and $\delta_U^V = \{p_U^+\}$, or $\delta_V^U = \{p_V^+\}$ and $\delta_U^V = \{p_U^-\}$.
(2) If $U \subsetneq V$ then $\mathsf{diam}(\delta_V^U) \leqslant E'$.
(3) If $V, W \subsetneq U$ are not transverse then $\mathsf{diam}_{T_U}(\delta_U^V \cup \delta_U^W) \leqslant E'$.
(4) If $U \pitchfork V$ and $W \subsetneq V$ and $U \not\perp W$ then $\mathsf{diam}_{T_U}(\delta_U^V \cup \delta_U^W) \leqslant E'$.
(5) If $U \perp V$ and $U, V \not\perp W$ and $W \not\sqsubseteq U, V$, then $\mathsf{diam}_{T_W}(\delta_W^U \cup \delta_W^V) \leqslant E'$.
(6) If $U \subsetneq V$ and C is a connected component of $T_V - \mathcal{N}_{E'}(\delta_V^U)$, then C contains p_V^+ or p_V^-. In the former case $\delta_U^V(C) = \{p_U^+\}$ and in the latter $\delta_U^V(C) = \{p_U^-\}$.
(7) If $V, W \subsetneq U$ and $V \pitchfork W$ and $d_{T_U}(\delta_U^V, \delta_U^W) > 10E'$ then $\delta_V^U(\delta_U^W) = \delta_V^W$ and $\delta_V^U(\delta_U^W) = \{p_V^+\}$ or $\{p_V^-\}$.

Proof. Suppose $U \pitchfork V$. By definition, there are sequences $(x_n^{\pm})_n$ in X such that $\pi_U(x_n^{\pm})$ converges to $\pi_U(p^{\pm})$ (or, if $\pi_U(p^{\pm}) \subseteq \mathcal{C}U$, to a uniformly bounded set that uniformly coarsely coincides with $\pi_U(p^{\pm})$. For all sufficiently large n, the consistency axiom implies that either

$$d_U(\rho_U^V, x_n^+), d_V(\rho_V^U, x_n^-) < E \quad \text{or} \quad d_U(\rho_U^V, x_n^-), d_V(\rho_V^U, x_n^+) < E,$$

so

$$d_U(\rho^V_U, \pi_U(p^+)), d_V(\rho^U_V, \pi_V(p^-)) < E \quad \text{or} \quad d_U(\rho^V_U, \pi_U(p^-)), d_V(\rho^U_V, \pi_V(p^+)) < E.$$

We assumed that $f_W(\mathcal{N}_{10(EM+E)}(\pi_W(p^\pm))) = p^\pm_W$ for $W \in \{U, V\}$, so this proves Item (1). Item (2) holds because f_V is an (L, L)-quasi-isometry and $\mathsf{diam}_U(\rho^U_V) \leqslant E$. Item (3) holds similarly because $\mathsf{diam}_U(\rho^V_U \cup \rho^W_U) \leqslant 2E$ by [DHS17, Lemma 1.5]. Item (4) is an immediate consequence of [BHS19, Defn. 1.1.(4)] and Item (5) follows from [DHS17, Lem. 1.5]. Item (6) follows from the bounded geodesic image axiom and the fact that f_U collapses the $10(EM + E)$-neighbourhoods of $\pi_U(p^\pm)$. Item (7) follows from Item (6) and the partial realisation axiom [BHS19, Defn. 1.1.(8)]. □

3.3. **The product of reduced trees $\widehat{\mathcal{Y}}$.** We now fix some constants, all depending only on the HHS parameters, to be used throughout the remainder of this section:

- $R := 50E'$;
- $\alpha_1 := 10R$;
- $\kappa := r_0(L\alpha_1 + LK + 2L)$, where r_0 is the constant from Theorem 2.15;
- $r := L\kappa + L + 1$.

Definition 3.11 (Clusters, edges and collapsed intervals)**.** Let $U \in \mathfrak{U}$. If U is $\sqsubseteq_{\mathfrak{U}}$-minimal then let $T^c_U := \varnothing$. Otherwise, we say a *basic cluster* in T_U is one of the following:

- the closed R–neighbourhood of some δ^V_U for which $V \subsetneq U$;
- one of the sets $\{p^-_U\}$ or $\{p^+_U\}$.

Declare two points $t, t' \in T_U$ to be equivalent if there are basic clusters c, c' such that $d_{T_U}(c, c') < r$ and $\{t, t'\}$ is contained in the convex hull in T_U of $c \cup c'$. Extend this to an equivalence relation by taking the transitive closure. A *cluster* in T_U is an equivalence class, and we define T^c_U to be the union of the clusters.

The components of $T^e_U := T_U - T^c_U$ are called *edge components*. Let $q_U : T_U \to \widehat{T}_U$ be the quotient map which collapses each cluster to a point. Let $\widehat{T}^c_U := q_U(T^c_U)$ and $\widehat{T}^e_U := q_U(T^e_U)$. ✽

Remark 3.12.

- By construction, clusters and edge components are subintervals of T_U.
- The quotient space $\widehat{T}_U$ is an interval, a line or a ray, and q_U restricts to an isometry on the edge components. ✽

Given $U \in \mathfrak{U}$, let $\widehat{p}^\pm_U := \lim_{n\to\infty} q_U(t_n)$ for some sequence $(t_n)_{n\in\mathbb{N}} \subseteq T_U$ which converges to $p^\pm_U$. Observe that this is independent of the choice of sequence, and if $p^\pm_U \in T_U$ then $\widehat{p}^\pm_U = q_U(p^\pm_U)$.

Remark 3.13.

(1) If $U \in \mathfrak{U}$ is $\sqsubseteq_{\mathfrak{U}}$-minimal then q_U is the identity map, and $\widehat{T}_U = T_U = T^e_U = \widehat{T}^e_U$.
(2) If $U \pitchfork V$ or $U \subsetneq V$ and $g \in G$, then $g\delta^U_V = \delta^{gU}_{gV}$.
(3) The above point implies that, for all $g \in G$ and $U \in \mathfrak{U}$, the isometry $g : T_U \to T_{gU}$ induces an isometry $g : \widehat{T}_U \to \widehat{T}_{gU}$ and we have $g \circ q_U = q_{gU} \circ g$. ✽

Let $\widehat{\mathcal{Y}} := \prod_{U\in\mathfrak{U}} \widehat{T}_U$ and define:

- the canonical projections $\widehat{\pi}_U : \widehat{\mathcal{Y}} \to \widehat{T}_U$ for all $U \in \mathfrak{U}$;
- if $U \pitchfork V$ or $U \subsetneq V$, then $\widehat{\delta}^U_V := q_V(\delta^U_V)$;
- if $U \subsetneq V$ then $\widehat{\delta}^V_U := q_U \circ \delta^V_U$.

By Remark 3.13.(3), the action of G on $\mathcal{Y}$ induces an action of G on $\widehat{\mathcal{Y}}$.

Notation 3.14. If $U \pitchfork V$ or $U \subsetneq V$ then $\widehat{\delta}_V^U$ is a singleton $\{x\} \subseteq \widehat{T}_V$ (by definition, in the nested case, and by the definition plus Lemma 3.10.(1) in the transverse case). We will abuse notation and refer to x itself as $\widehat{\delta}_V^U$. ✽

The following is a consequence of Lemma 3.10 and the choice of constant R.

Lemma 3.15. *For all $U, V, W \in \mathfrak{U}$ the following hold:*

(1) If $U \pitchfork V$ then either $\widehat{\delta}_V^U = \widehat{p}_V^+$ and $\widehat{\delta}_U^V = \widehat{p_U^-}$, or $\widehat{\delta}_V^U = \widehat{p_V^-}$ and $\widehat{\delta}_U^V = \widehat{p}_U^+$.
(2) If $V, W \subsetneq U$ are not transverse then $\widehat{\delta}_U^V = \widehat{\delta}_U^W$.
(3) If $U \pitchfork V$ and $W \subsetneq V$ and $U \not\perp W$ then $\widehat{\delta}_U^V = \widehat{\delta}_U^W$.
(4) If $V \subsetneq U$ and C^- and C^+ are the connected components of $\widehat{T}_U - \widehat{\delta}_U^V$ containing $\widehat{p_U^-}$ and p_U^+ respectively, then $\widehat{\delta}_V^U(C^\pm) = \widehat{p}_V^\pm$.

3.4. **The median space Q.** We now construct the median space Q whose existence is asserted in Theorem 3.1.

Definition 3.16 (Consistency, the space Q)**.** A tuple $(x_U)_{U \in \mathfrak{U}} \in \widehat{\mathcal{Y}}$ is *consistent* if, for all $U, V \in \mathfrak{U}$,

- if $U \pitchfork V$, then $x_U = \widehat{\delta}_U^V$ or $x_V = \widehat{\delta}_V^U$ and
- if $U \subsetneq V$ then either $x_V = \widehat{\delta}_V^U$ or $x_U \in \widehat{\delta}_U^V(x_V)$.

Let $Q \subseteq \widehat{\mathcal{Y}}$ be the set of consistent tuples and define $d_Q(x, y) := \sum_{U \in \mathfrak{U}} d_{\widehat{T}_U}(\widehat{\pi}_U(x), \widehat{\pi}_U(y))$ for all $x, y \in Q$. ✽

Remark 3.13.(2) and Lemma 3.15.(4) imply:

Lemma 3.17. *The subspace $Q \subseteq \widehat{Y}$ is G–invariant.*

Definition 3.18 (Relevant domains)**.** Given $x, y \in Q$, let

$$\mathrm{Rel}_Q(x, y) := \{U \in \mathfrak{U} : \widehat{\pi}_U(x) \neq \widehat{\pi}_U(y)\}. \quad ✽$$

We now establish that d_Q defines a median metric on Q. Recall that χ denotes the complexity of $(X, \mathfrak{S})$.

Lemma 3.19.

(1) There exists χ', depending only on χ, such that, if $x \in Q$ then for all but at most χ' domains $U \in \mathfrak{U}$ we have $x_U \in \{\widehat{p_U^-}, \widehat{p}_U^+\}$.
(2) For all $\widehat{x}, \widehat{y} \in Q$ the set $\mathrm{Rel}_Q(\widehat{x}, \widehat{y})$ is finite.
(3) (Q, d_Q) is a median metric space with rank at most χ.

Proof.

(1) Let $x \in Q$, and let $\mathfrak{U}_x := \{V \in \mathfrak{U} : x_V \notin \{p_V^\pm\}\}$. If $U, V \in \mathfrak{U}_x$ then $U \perp V$ or U and V are $\sqsubseteq$-related, by consistency of x and Lemma 3.15. Let Γ be the complete graph with vertex set $\mathfrak{U}_x$. Given $U, V \in \mathfrak{U}_x$, colour the edge between them red if $U \perp V$ and blue if they are $\sqsubseteq$-related. Blue cliques have at most χ vertices by the definition of χ and red cliques have at most χ vertices by [BHS19, Lemma 2.1]. Thus Ramsey's theorem provides the required χ'.

(2) If $p^+, p^- \in X$ then $\mathrm{Rel}_Q(\widehat{x}, \widehat{y}) \subseteq \mathfrak{U} \subseteq \mathrm{Rel}_{K/L-L}(p^+, p^-)$, which is finite by [CRHK24, Lemma 13.5] and the choice of K.

Otherwise, suppose, without loss of generality, that $p^+ \in \partial X$. For each $U \in \mathfrak{U}$, let $x_U \in \mathcal{C}U$ be such that $d_{T_U}(f_U(x_U), \widehat{x}_U) \leqslant L$, and define y_U analogously. For each $V \in \mathrm{supp}(p^+)$, let $W_V := \{U \in \mathrm{Rel}_Q(\widehat{x}, \widehat{y}) : U \subsetneq V\}$. Then there is a uniformly bounded neighbourhood of $\mathrm{hull}_{\mathcal{C}V}(x_V, y_V)$ which contains ρ_V^U for all $U \in W_V$. Therefore there exists a point $z_V^+ \in \mathcal{C}V$ such that z_V^+ is in the same connected component of

$\mathcal{C}V - \mathcal{N}_{2E+\mathsf{Morse}_E(1,20E)}(\rho^U_V)$ as $\pi_V(p^+)$ for all $U \in W_V$. Using partial realisation (Definition [BHS19, Defn. 1.1.(8)]), let $z^+ \in X$ be a point such that $d_V(z^+, z^+_V) \leqslant E$ for all $V \in \operatorname{supp}(p^+)$. If $p^- \in X$ then $\operatorname{Rel}_Q(\widehat{x},\widehat{y}) \subseteq \operatorname{Rel}_{K/L+L}(z^+,p^-)$. If not, then we can define a point $z^- \in X$ analogously to z^+ so that $\operatorname{Rel}_Q(\widehat{x},\widehat{y}) \subseteq \operatorname{Rel}_{K/L+L}(z^+,z^-)$. In either case, [CRHK24, Lemma 13.5] implies that $\operatorname{Rel}_Q(\widehat{x},\widehat{y})$ is finite.

(3) The fact that d_Q is a metric follows easily from Item (2). For each $U \in \mathfrak{U}$, let m_U denote the median map on $\widehat{T}_U$. It is straightforward to check that $m = (m_U)_{U\in\mathfrak{U}}$ is consistent and that it is the metric median map for (Q, d_Q). If $C = \{\sigma(\varepsilon_1,\dots,\varepsilon_k) : \varepsilon_i \in \{0,1\}\} \subseteq Q$ is a median embedded k-cube, then consistency implies the following: let $x,y,z \in C$ be such that x differs from each of y and z in exactly one coordinate, and y,z differ in two coordinates. Then $U \perp V$ whenever $U \in \operatorname{Rel}_Q(x,y)$ and $V \in \operatorname{Rel}_Q(x,z)$. Hence $\mathfrak{U}$ contains a set of k pairwise orthogonal elements, so $k \leqslant \chi$. (The consistency argument for the rank bound is virtually identical to the proof of [CRHK24, Lem. 6.13], for instance, so we omit the details.) □

Lemma 3.20. *Q is complete.*

Proof. Let $(x^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Q. For each $U \in \mathfrak{U}$, the sequence $(x^n_U)_{n\in\mathbb{N}} \subseteq \widehat{T}_U$ is Cauchy. Let $x_U \coloneqq \lim_{n\to\infty} x^n_U$ and let $x \coloneqq (x_U)_{U\in\mathfrak{U}}$. It follows from the consistency of the x^n's that x is consistent. To conclude that $\lim_{n\to\infty} x^n = x$, it then suffices to show that $|\cup_{n\in\mathbb{N}} \operatorname{Rel}_Q(x^1,x^n)| < \infty$. If not, another application of Ramsey's theorem will produce, for all $N \in \mathbb{N}$, a set $\mathfrak{V}$ of pairwise transverse domains such that $|\mathfrak{V}| \geqslant N$ and $V \subseteq \operatorname{Rel}_Q(x^1,x^n)$ for all sufficiently large n. By Lemma 3.19.(1) we can remove $\leqslant \chi'$ elements from $\mathfrak{V}$ and assume that $x^1_U = p^-_U$ and $x^n_U = p^+_U$ for all $U \in \mathfrak{V}$ (or vice versa). By Lemma 3.10.(6), we can also assume that each $U \in \mathfrak{V}$ is $\sqsubseteq_{\mathfrak{U}}$–minimal, which implies that $d_{\widehat{T}_U}(p^-_U, p^+_U) \geqslant K$ for all $U \in \mathfrak{U}$. But then $d_Q(x,x^n) \to \infty$. □

Lemma 3.21. *Q is path-connected.*

Proof. Let $x,y \in Q$ be distinct elements. We will argue by induction on the cardinality of $\operatorname{Rel}_Q(x,y)$ that there is a path in Q connecting x and y.

Suppose $\operatorname{Rel}_Q(x,y) = \{U\}$ and let $\gamma_U : [0,1] \to \widehat{T}_U$ be the unique geodesic from $\widehat{\pi}_U(x)$ to $\widehat{\pi}_U(y)$. For each $t \in [0,1]$, let $z_t = (z_V)_{V\in\mathfrak{U}}$, where $z_V \coloneqq x_V = y_V$ for all $V \neq U$ and $z_U \coloneqq \gamma_U(t)$. It follows from the fact that both x and y are consistent that z_t is consistent. Thus we can define $\gamma : [0,1] \to Q$ by $\gamma(t) \coloneqq z_t$ for all $t \in [0,1]$. Continuity of γ follows immediately from continuity of γ_U and the definition of d_Q, so γ is a path.

Let $k \in \mathbb{N}$, suppose that $|\operatorname{Rel}_Q(x,y)| = k+1$ and suppose x' and y' are connected by a path for all $x',y' \in Q$ such that $|\operatorname{Rel}_Q(x',y')| \leqslant k$. We consider two cases:

Case 1: Suppose there exist $U,V \in \operatorname{Rel}_Q(x,y)$ such that $V \subsetneq U$. We can assume without loss of generality that U is $\sqsubseteq$-maximal in $\operatorname{Rel}_Q(x,y)$. Define $z^1 = (z^1_W)_{W\in\mathfrak{U}}, z^2_W = (z^2_W)_{W\in\mathfrak{U}}$ as follows. For each $W \in \mathfrak{U}$ such that $W \sqsubseteq V$ or $W \perp V$, let $z^1_W \coloneqq x_W$ and let $z^2_W \coloneqq y_W$. For each $W \in \mathfrak{U}$ such that $W \pitchfork V$ or $V \subsetneq W$, let $z^1_W \coloneqq z^2_W \coloneqq \widehat{\delta}^V_W$.

Let us check that z^1 and z^2 are consistent. Let $W_1, W_2 \in \mathfrak{U}$ be transverse. If W_1, W_2 are either nested in or orthogonal to V, then the required condition follows from the consistency of x and y. If $W_1 \sqsubseteq V$ or $W_1 \perp V$, and $W_2 \not\sqsubseteq V$ and $W_2 \not\perp V$, then $\widehat{\delta}^{W_1}_{W_2} = \widehat{\delta}^V_{W_2} = z^1_{W_2} = z^2_{W_2}$. Suppose W_1 and W_2 are neither nested in nor orthogonal to V and $\widehat{\delta}^V_{W_1} \neq \widehat{\delta}^{W_2}_{W_1}$. Then $d_{W_1}(\rho^V_{W_1}, \rho^{W_2}_{W_1}) > r/L - L > K_2 + 10E$. Since $\mathsf{diam}(\mathcal{C}V) > 100E$ and π_V is E–coarsely surjective, there exists $p \in X$ such that $d_V(p, \pi_V(p^{\pm})) > 10E$, which implies by the consistency and bounded geodesic image axioms that $d_{W_1}(p, \rho^V_{W_1}) < E + K_2$. Then

$d_{W_1}(p, \rho^{W_2}_{W_1}) > E$ so $d_{W_2}(p, \rho^{W_1}_{W_2}), d_{W_2}(p, \rho^{V}_{W_2}) \leqslant E$. Thus $d_{W_2}(\rho^{W_1}_{W_2}, \rho^{V}_{W_2}) \leqslant 2E$, which implies that $\widehat{\delta}^{W_1}_{W_2} = \widehat{\delta}^{V}_{W_2}$. Next suppose that $W_1 \subsetneq W_2$. Again, if $W_2 \subsetneq V$ or $W_2 \perp V$, then the required condition follows from the consistency of x and y. Suppose $V \subsetneq W_2$. If $\widehat{\delta}^{W_1}_{W_2} \neq \widehat{\delta}^{V}_{W_2}$ then Lemma 3.15.(2,4), gives $W_1 \pitchfork V$ and $\widehat{\delta}^{V}_{W_1} = \widehat{\delta}^{W_2}_{W_1}(\widehat{\delta}^{V}_{W_2})$. Lastly, suppose $V \pitchfork W_2$ and without loss of generality, suppose that $\widehat{\delta}^{V}_{W_2} = \widehat{p}^{+}_{W_2}$. Then either $\widehat{\delta}^{W_1}_{W_2} = \widehat{\delta}^{V}_{W_2}$ or $V \pitchfork W_1$ and $\widehat{\delta}^{W_2}_{W_1}(\widehat{p}^{+}_{W_2}) = \widehat{p}^{+}_{W_1} = \widehat{\delta}^{V}_{W_1}$ by Lemma 3.15.(4).

Observe that $|\operatorname{Rel}_Q(x, z^1)|, |\operatorname{Rel}_Q(z^1, z^2)|, |\operatorname{Rel}_Q(z^2, y)| \leqslant k$ so there exist paths in Q connecting x to z^1, z^1 to z^2 and z^2 to y. Their concatenation is a path in Q from x to y.

Case 2: If Case 1 fails then for all U, V in $\operatorname{Rel}_Q(x, y)$, either $U \pitchfork V$ or $U \perp V$. Then, up to swapping x and y, we can write $\operatorname{Rel}_Q(x, y) = \{U_0, \ldots, U_k\}$, where the labels are chosen so that, if $0 < j$, then $U_0 \perp U_j$ or $\widehat{\delta}^{U_j}_{U_0} = \widehat{p}^{+}_{U_0} = y_{U_0}$ and $\widehat{\delta}^{U_0}_{U_j} = \widehat{p}^{-}_{U_j} = x_{U_j}$; this is a consequence of consistency (compare [BHS19, Prop. 2.8]). Define $z = (z_W)_{W \in \mathfrak{U}}$ where $z_W := x_W$ for all $W \neq U_0$ and $z_{U_0} := y_{U_0}$. It is straightforward to check that z is consistent. Moreover $|\operatorname{Rel}_Q(x, z)| = 1$ and $|\operatorname{Rel}_Q(z, y)| = k$ so the induction hypothesis implies that there are paths in Q from x and z and from z to y, whose concatenation is the required path in Q. $\square$

Lemma 3.22. *The space (Q, d_Q) isometrically embeds in $(\mathbb{R}^k, \ell_1)$ for some k, and hence the space (Q, d_Q) is proper.*

Proof. Given $U, V \in \mathfrak{U}$, write $U < V$ if $U \pitchfork V$ and $\widehat{\delta}^{U}_{V} = \widehat{p}^{-}_{V}$ (and hence $\widehat{\delta}^{V}_{U} = \widehat{p}^{+}_{U}$, by Lemma 3.15). Note that $<$ is a partial order on $\mathfrak{U}$, and $U, V \in \mathfrak{U}$ are incomparable in this order if and only if they are either orthogonal or $\subsetneq$–related. Hence, by Ramsey's theorem, there is a bound (depending only on the complexity of $\mathfrak{S}$) on the cardinalities of $<$–antichains in $\mathfrak{U}$. Therefore, by Dilworth's theorem [Dil50], there exists a finite partition $\mathfrak{U} = \bigsqcup_{i=1}^{k} \mathfrak{U}^i$ where each $\mathfrak{U}^i$ is a $<$–chain. For each $U \in \mathfrak{U}^i$, choose $x(U) \in Q$ such that $x_U \notin \{\hat{p}^{\pm}_U\}$ $U \in \mathfrak{U}_{x(U)}$, using Lemma 3.21.

By Lemma 3.19.(2), $\operatorname{Rel}_Q(x(U), x(V))$ is finite for all $U, V \in \mathfrak{U}^i$, so the totally ordered set $(\mathfrak{U}^i, <)$ has finite intervals, so there is an interval $I \subseteq \mathbb{R}$ such that $\mathfrak{U}^i = \{U^n : n \in \mathbb{Z} \cap I\}$, where the U^n are labelled so that $n < m$ if and only if $U^n < U^m$. Note that, if U^n has a successor (predecessor), then the interval $\widehat{T}_{U^n}$ has a maximum (minimum). Hence there is a space L^i, isometric to $\mathbb{R}, [0, \infty)$, or a bounded interval, equipped with isometric embeddings $\iota^n : \widehat{T}_{U^n} \to L^i$ such that $\bigcup_n \iota^n(\widehat{T}_{U^n}) = L^i$, and ι^m, ι^n have disjoint images unless $|m - n| \leqslant 1$, and $\iota^n(\widehat{T}_{U^n}) \cap \iota^{n+1}(\widehat{T}_{U^{n+1}}) = \iota^n(p^{+}_{U^n}) = \iota^{n+1}(p^{-}_{U^{n+1}})$. Define $\pi^i : Q \to L^i$ as follows. Given $x \in Q$, let n be maximal such that $x_{U^n} = p^{+}_{U^n}$. If $n + 1 \notin I$, then L^i has a maximum, which we take as $\pi^i(x)$. If $n + 1$ exists, we let $\pi^i(x) = \iota^{n+1}(x_{U^{n+1}})$. If there is no such n, then there is a unique minimal $n \in I \cap \mathbb{Z}$, and we take $\pi^i(x) = \iota^n(x_{U^n})$. Hence $\prod_{i=1}^{k} \pi^i : Q \to \prod_{i=1}^{k} L_i$ is an isometric embedding (where the codomain has the ℓ_1 metric), so Q is locally compact and hence, since it is complete by Lemma 3.20, proper. $\square$

3.5. **Defining the map $\widehat{\Psi} : X \to Q$.** For each $U \in \mathfrak{U}$, let $\psi_U := f_U \circ \pi_U$. Then define $\Psi : X \to \mathcal{Y}$ by $\Psi(x) := (\psi_U(x))_{U \in \mathfrak{U}}$. By Definition 2.10 and Proposition 3.6, the map Ψ is G-equivariant.

Definition 3.23. Let $\alpha \geqslant 0$. An element $x = (x_U)_{U \in \mathfrak{U}} \in \mathcal{Y}$ is *α-consistent* if, for all $U, V \in \mathfrak{U}$,

- if $U \pitchfork V$ then $\min\{d_{T_U}(x_U, \delta^V_U), d_{T_V}(x_V, \delta^U_V)\} \leqslant \alpha$;
- if $U \subsetneq V$ then $\min\{d_{T_V}(x_V, \delta^U_V), \mathsf{diam}(\{x_U\} \cup \delta^V_U(x_V))\} \leqslant \alpha$.

Let $\mathcal{Z}_\alpha$ be the set of α-consistent tuples in $\mathcal{Y}$. ✽

Next is an elementary consequence of Lemma 3.10 and the HHS consistency axioms:

Lemma 3.24. *We have $\Psi(x) \in \mathcal{Z}_\alpha$ for all $x \in X$ and $\alpha \geqslant E'$.*

Define $\Phi : \mathcal{Y} \to \widehat{\mathcal{Y}}$ by $(x_U)_U \mapsto (q_U(x_U))_U$, and let $\widehat{\Psi} := \Phi \circ \Psi$. For each $U \in \mathfrak{U}$, also define $\widehat{\psi}_U : X \to \widehat{T}_U$ by $\widehat{\psi}_U := q_U \circ \psi_U$. By construction, $\widehat{\Psi}$ is G–equivariant.

Lemma 3.25. *The image of $\widehat{\Psi}$ is consistent in the sense of Definition 3.16, i.e. $\widehat{\Psi}(X) \subseteq Q$.*

Proof. Let $x \in X$ and $U, V \in \mathfrak{U}$. If $U \pitchfork V$ then $\min\{d_{T_U}(\psi_U(x), \delta_U^V), d_{T_V}(\psi_V(x), \delta_V^U)\} \leqslant E' < R$ so either $\widehat{\psi}_U(x) = \widehat{\delta}_U^V$ or $\widehat{\psi}_V(x) = \widehat{\delta}_V^U$. Suppose $U \subsetneq V$. If $\widehat{\psi}_V(x) \neq \widehat{\delta}_V^U$, then $d_{T_V}(\delta_V^U, \psi_V(x)) > R > 50E'$ so $d_V(\rho_V^U, x) > 50E$. Therefore $\mathsf{diam}_{\mathcal{C}U}(\pi_U(x) \cup \rho_U^V(\pi_V(x))) \leqslant E$ by the consistency axiom for HHSes ([BHS19, Defn. 1.1.(4)]). It follows from the bounded geodesic image axiom for HHSes ([BHS19, Defn. 1.1.(7)]) that

$$\min\{\mathsf{diam}_{\mathcal{C}U}(\rho_U^V(\pi_V(x)) \cup \pi_U(p^-)), \mathsf{diam}_{\mathcal{C}U}(\rho_U^V(\pi_V(x)) \cup \pi_U(p^+))\} \leqslant 5E,$$

so $\psi_U(x) = p_U^+$ or p_U^- and $\psi_U(x) \in \delta_U^V(\psi_V(x))$. Thus $\widehat{\psi}_U(x) \in \widehat{\delta}_U^V(\widehat{\psi}_V(x))$, as required. □

3.6. **Coarse surjectivity.** Next we check that X coarsely surjects to Q, by proving:

Proposition 3.26. *There exists M_1, depending only on the HHS parameters, such that $\widehat{\Psi}$ is M_1-coarsely surjective.*

We first need the following lemma; recall that $\alpha_1 = 10R$.

Lemma 3.27. *The map $\Phi|_{\mathcal{Z}_{\alpha_1}} : \mathcal{Z}_{\alpha_1} \to Q$ is surjective.*

Proof. Let $\widehat{x} = (\widehat{x}_U)_{U \in \mathfrak{U}} \in Q$. Let us construct an α_1–consistent tuple $x = (x_U)_{U \in \mathfrak{U}} \in \mathcal{Y}$ such that $\Phi(x) = \widehat{x}$. Let $U \in \mathfrak{U}$. If $\hat{x}_U \in \widehat{T}_U^e$, then $q_U^{-1}(\hat{x}_U)$ contains a single point, which we denote x_U. Otherwise, let $C \subseteq T_U$ be the cluster such that $q_U(C) = \widehat{x}_U$. Define

$$\mathfrak{U}_C := \{V \in \mathfrak{U} : V \subsetneq U,\ \delta_U^V \subseteq C \text{ and } \widehat{x}_V \in \widehat{T}_V^e\}.$$

Note that $\mathfrak{U}_C$ contains at least all $\sqsubseteq_{\mathfrak{U}}$-minimal $V \sqsubseteq U$ such that $\delta_U^V \subseteq C$, by Remark 3.13. For each $V \in \mathfrak{U}_C$, we have a unique point $x_V \in q_V^{-1}(\widehat{x}_v)$. Given $V \in \mathfrak{U}_C$, define $C_V \subseteq T_U$ as follows. If $x_V \notin \{p_V^-, p_V^+\}$ then define $C_V := \mathcal{N}_R(\delta_U^V)$. If $x_V = p_V^-$ then let $C_V := C \cap \mathrm{hull}_{T_U}(\{p_U^-\} \cup \mathcal{N}_R(\delta_U^V))$ and if $x_V = p_V^+$ then let $C_V := C \cap \mathrm{hull}_{T_U}(\mathcal{N}_R(\delta_U^V) \cup \{p_U^+\})$.

Claim 1. If $V, W \in \mathfrak{U}_C$ then $C_V \cap C_W \neq \varnothing$.

Proof. Suppose $V \neq W$. If V and W are not transverse then Lemma 3.10.(3) implies that $d_{T_U}(\delta_U^V, \delta_U^W) < R$, so $C_V \cap C_W \neq \varnothing$ by construction. If $V \pitchfork W$, then consistency of $\widehat{x}$ implies that either $x_V \in \delta_V^W \in \{p_V^\pm\}$ or $x_W \in \delta_W^V \in \{p_W^\pm\}$. Suppose $x_V = p_V^+$; the other cases are similar. If $d_{T_U}(\rho_U^V, \rho_V^W) > R > 10E'$ then Lemma 3.10.(6,7) imply that δ_U^W is in the same connected component of $T_U - \mathcal{N}_R(\delta_U^V)$ as p_U^+, so $\delta_U^W \subseteq C_V \cap C_W$. ■

Identify T_U isometrically with a subset of $\mathbb{R}$ such that, if $p_U^+, p_U^- \in T_U$ then $p_U^- < p_U^+$ and, if $p_U^+ \in \partial T_U$ (resp. $p_U^- \in \partial T_U$) then, extending the identification to the boundary, $p_U^+ = +\infty$ (resp. $p_U^- = -\infty$).

We define $x_U \in T_U$ as follows, distinguishing two cases. First, if $U \notin \mathrm{supp}(p^-) \cup \mathrm{supp}(p^+)$, then T_U is a finite interval. Claim 1 implies that $\cap_{V \in \mathfrak{U}_C} C_V \neq \varnothing$. In this case, choose $x_U \in \cap_{V \in \mathfrak{U}_C} C_V$ such that:

- if $x_V = p_V^+$ for all $V \in \mathfrak{U}_C$ then $x_U := \sup C$;
- if $x_V = p_V^-$ for all $V \in \mathfrak{U}_C$ then $x_U := \inf C$;
- otherwise, there exists $W \in \mathfrak{U}_C$ such that $x_W \notin \{p_W^\pm\}$, so $C_W = \mathcal{N}_R(\delta_U^W)$, and we choose $x_U \in \cap_{V \in \mathfrak{W}} C_V \subseteq \mathcal{N}_R(\delta_U^W)$ arbitrarily.

Since T_U is a finite interval, $x_U \in T_U$ is well defined.

The second case is where $U \in \operatorname{supp}(p^+) \cup \operatorname{supp}(p^-)$; without loss of generality, suppose $U \in \operatorname{supp}(p^+)$. If there exists $W \in \mathfrak{U}_C$ such that $x_W \notin \{p_W^\pm\}$, then, exactly as in the first case, $C_W = \mathcal{N}_R(\delta_U^W)$, and we choose $x_U \in \cap_{V \in \mathfrak{W}} C_V \subseteq \mathcal{N}_R(\delta_U^W)$ arbitrarily. Hence we can assume that $x_V = p_V^+$ for all $V \in \mathfrak{U}_C$ (the case where $x_V = p_V^-$ for all $V \in \mathfrak{U}_C$ is similar).

Suppose $\sup C = p_U^+$, so, by the definition of a cluster, $\sup(\cup_{V \in \mathfrak{U}_C} \delta_U^V) = p_U^+$. (So, $p_U^+ \in \partial T_U$ but $\max \widehat{T}_U = \hat{x}_U$.) Hence there is a sequence $\{V_n\}_{n \in \mathbb{N}}$ of $\sqsubseteq_{\mathfrak{U}}$–minimal elements in $\mathfrak{U}_C$ such that $V_n \pitchfork V_m$ and $\delta_{V_n}^{V_m} = p_{V_n}^+$ and $\delta_{V_m}^{V_n} = p_{V_m}^-$ whenever $n < m$, and $\lim_{n\to\infty}(\sup \delta_U^{V_n}) = p_U^+$. By our assumption, $x_{V_n} = p_{V_n}^+$ for all n. Now, for any $z \in X$, we have $\widehat{\psi}_{V_n}(z) = \widehat{p}_{V_n}^-$ for all but finitely many n. By Lemma 3.25, $\widehat{\Psi}(z) \in Q$, but we have just shown that $\operatorname{Rel}_Q(\widehat{\Psi}(z), \hat{x})$ is infinite, contradicting Lemma 3.19. Thus $\sup C < \infty$ and we define $x_U := \sup C$.

Let $x = (x_U)_{U \in \mathfrak{U}} \in \mathcal{Y}$. By construction, $\Phi(x) = \hat{x}$, so it remains to show that x is $10R$-consistent.

Let $U_1, U_2 \in \mathfrak{U}$ be such that $U_1 \sqsubsetneq U_2$. Suppose first that $\hat{x}_{U_2} \neq \widehat{\delta}_{U_2}^{U_1}$, so $\hat{x}_{U_1} \in \widehat{\delta}_{U_1}^{U_2}(\hat{x}_{U_2}) = \{\widehat{p}_{U_1}^-\}$ or $\{\widehat{p}_{U_1}^+\}$. Suppose the latter holds; the other case is similar. We then have $d_{U_2}(x_{U_2}, \delta_{U_2}^{U_1}) > R > E'$ and x_{U_2} is in the same connected component of $T_{U_2} - \mathcal{N}_{E'}(\delta_{U_2}^{U_1})$ as $p_{U_2}^+$, so Lemma 3.10.(6) implies that $\delta_{U_1}^{U_2}(x_{U_2}) = p_{U_1}^+$. If $x_{U_1} = \hat{x}_{U_1}$, then this implies immediately that $\delta_{U_1}^{U_2}(x_{U_2}) = x_{U_1}$. Otherwise, let $C \subseteq T_{U_1}$ be the cluster containing x_{U_1}. Then, for each domain $V \in \mathfrak{U}_C$, we have $\mathsf{diam}_{T_{U_2}}(\delta_{U_2}^V \cup \delta_{U_2}^{U_1}) < R$ by Lemma 3.10.(3) so $\widehat{\delta}_{U_2}^V = \widehat{\delta}_{U_2}^{U_1} \neq \hat{x}_{U_2}$. By consistency of $\hat{x}$, we have $\hat{x}_V \in \widehat{\delta}_V^{U_2}(\hat{x}_{U_2}) = \{p_V^+\}$ which, since $V \in \mathfrak{U}_C$, implies that $x_V = p_V^+$. Then $x_{U_1} = \rho_{U_1}^+ = \delta_{U_1}^{U_2}(x_{U_2})$.

Next suppose that $\hat{x}_{U_2} = \widehat{\delta}_{U_2}^{U_1}$ and let $C \subseteq T_{U_2}$ be the cluster containing $\delta_{U_2}^{U_1}$ and x_{U_2}. Suppose $d_{T_{U_2}}(x_{U_2}, \delta_{U_2}^{U_1}) > 2R$. Then Lemma 3.10.(6) implies that, either $x_V = p_V^+$ for all $V \in \mathfrak{U}_C$ such that $V \sqsubseteq U_1$, or $x_V = p_V^-$ for all $V \in \mathfrak{U}_C$ such that $V \sqsubseteq U_1$. Suppose the former case holds; the latter case is similar. If $\hat{x}_{U_1} \notin \widehat{T}_{U_1}^e$, then $x_{U_1} = p_{U_1}^+$ by construction and, if $\hat{x}_{U_1} \in \widehat{T}_{U_1}^e$, then $U_1 \in \mathfrak{U}_C$ so $x_{U_1} = p_{U_1}^+$. Moreover $x_{U_2} \in \cap_{V \in \mathfrak{U}_C} C_V - \mathcal{N}_{2R}(\delta_{U_2}^{U_1})$, which implies that $\delta_{U_1}^{U_2}(x_{U_2}) = p_{U_1}^+$.

Now let $U_1, U_2 \in \mathfrak{U}$ be such that $U_1 \pitchfork U_2$. Up to relabelling, we can assume that $\delta_{U_2}^{U_1} = \{p_{U_2}^-\}$ and $\delta_{U_1}^{U_2} = \{p_{U_1}^+\}$, so we have $\hat{x}_{U_1} = \widehat{p}_{U_1}^+$ or $\hat{x}_{U_2} = \widehat{p}_{U_2}^-$. We first suppose that $\hat{x}_{U_1} = \widehat{p}_{U_1}^+$ and $\hat{x}_{U_2} \neq \widehat{p}_{U_2}^-$. If $\hat{x}_{U_1} \in \widehat{T}_{U_1}^e$, then $x_{U_1} = p_{U_1}^+ \in \delta_{U_1}^{U_2}$. Otherwise, $p_{U_1} \in C$ for some cluster $C \subseteq T_{U_1}$ such that $\mathfrak{U}_C \neq \emptyset$. Let $V \in \mathfrak{U}_C$. If $V \pitchfork U_2$ then $x_V = \hat{x}_V = \widehat{p}_V^+ = p_V^+$. Otherwise $V \perp U_2$ and Lemma 3.10.(5) implies that $\mathsf{diam}(\{p_{U_1}^+\} \cup \delta_{U_1}^V) = \mathsf{diam}(\delta_{U_1}^{U_2} \cup \delta_{U_1}^V) \leqslant E' \leqslant R$. It follows that $x_{U_1} \in \mathcal{N}_R(p_{U_1}^+)$. If $\hat{x}_{U_1} \neq \widehat{p}_{U_1}^+$ and $\hat{x}_{U_2} = \widehat{p}_{U_2}^-$ then a symmetric argument shows that $d_{T_{U_2}}(x_{U_2}, p_{U_2}^-) \leqslant R$.

Suppose that $\hat{x}_{U_1} = \widehat{p}_{U_1}^+$ and $\hat{x}_{U_2} = \widehat{p}_{U_2}^-$. Suppose that $x_{U_1} \neq p_{U_1}^+$ and $x_{U_2} \neq p_{U_2}^-$. Then there exist clusters $C_1 \subseteq T_{U_1}, C_2 \subseteq T_{U_2}$, containing $p_{U_1}^+, p_{U_2}^-$ respectively, such that $\mathfrak{U}_{C_1}, \mathfrak{U}_{C_2} \neq \emptyset$. If $\hat{x}_V = \widehat{p}_V^+$ for all $V \in \mathfrak{U}_{C_1}$ then $x_{U_1} = p_{U_1}^+$ and, if $\hat{x}_V = \widehat{p}_V^-$ for all $V \in \mathfrak{U}_{C_2}$ then $x_{U_2} = p_{U_2}^-$. So there are $V_1 \in \mathfrak{U}_{C_1}$ and $V_2 \in \mathfrak{U}_{C_2}$ with $\hat{x}_{V_1} \neq \widehat{p}_{V_1}^+$ and $\hat{x}_{V_2} \neq \widehat{p}_{V_2}^-$.

Suppose $V_1 \perp V_2$. If $V_2 \not\perp U_1$ then Lemma 3.10.(4) implies that $\mathsf{diam}(\delta_{U_1}^{V_2} \cup \{p_{U_1}^+\}) = \mathsf{diam}(\delta_{U_1}^{V_2} \cup \delta_{U_1}^{U_2}) \leqslant E'$ and $\mathsf{diam}(\delta_{U_1}^{V_1} \cup \delta_{U_1}^{V_2}) \leqslant E'$ so $\mathsf{diam}(\delta_{U_1}^{V_1} \cup \{p_{U_1}^+\}) < R$. If $V_2 \perp U_1$ then Lemma 3.10.(5) implies that $\mathsf{diam}(\delta_{U_2}^{V_2} \cup \delta_{U_2}^{U_1}) \leqslant E'$.

By construction, $d_{T_{U_2}}(x_{U_2}, \delta_{U_2}^{V_2}) \leqslant R$, so

$$d_{T_{U_2}}(x_{U_2}, \delta_{U_2}^{U_1}) \leqslant E' + R < 10R.$$

If $V_1 \not\perp V_2$ then, since $\widehat{x}$ is consistent, V_1 and V_2 are $\sqsubseteq$-related, so $V_i \sqsubseteq U_1$ and $V_i \sqsubseteq U_2$ for some $i \in \{1,2\}$. Hence $d_{U_1}(\rho_{U_1}^{V_i}, \rho_{U_1}^{U_2}) \leqslant E$ so $\delta_{U_1}^{V_i} = p_{U_1}^{+} = \delta_{U_1}^{U_2}$. Then $x_{U_1} \in \mathcal{N}_R(\delta_{U_1}^{V_1}) \subseteq \mathcal{N}_{2R}(\delta_{U_1}^{U_2})$ by construction and Lemma 3.10. □

Proof of Proposition 3.26. For later use, fix $z \in X$ and, for all $V \in \mathfrak{S} - \mathfrak{U}$, let $z_V := f_V(\pi_V(z))$.

Let $\widehat{x} \in Q$; we need to find $y \in X$ with $d_Q(x, \widehat{\Psi}(y)) \leqslant M_1$, where M_1 is uniform.

Using Lemma 3.27, let $x \in \mathcal{Z}_{\alpha_1}$ be such that $\Phi(x) = \widehat{x}$. For each $U \in \mathfrak{U}$, there exists $y_U \in \mathcal{C}U$ such that $d_{T_U}(f_U(y_U), x_U) \leqslant L$. If $U \in \mathfrak{S} - \mathfrak{U}$ then let $y_U := z_U$. The fact that x is α_1-consistent and $\mathsf{diam}(\mathcal{C}U) \leqslant LK + L$ for all $U \in \mathfrak{S} - \mathfrak{U}$ implies that $y = (y_U)_{U \in \mathfrak{S}}$ is $(L\alpha_1 + LK + 2L)$-consistent. Hence, by Theorem 2.15 and the definition of κ, there exists $y \in X$ such that $d_U(y, y_U) \leqslant \kappa$ for all $U \in \mathfrak{U}$. Thus $d_{\widehat{T}_U}(\widehat{\psi}_U(y), \widehat{x}_U) \leqslant d_{T_U}(\psi_U(y), x_U) \leqslant L\kappa + L$. By Lemma 3.19.(1),

$$|\{V \in \mathrm{Rel}_Q(\widehat{x}, \widehat{\Psi}(y)) : \{\widehat{x}_V, \widehat{\psi}_V(y)\} \neq \{\widehat{p}_V^{\pm}\}\}| \leqslant \chi'.$$

Let $\mathcal{V}$ be the set of elements in $\mathrm{Rel}_Q(\widehat{x}, \widehat{\Psi}(y))$ such that $\{\widehat{x}_V, \widehat{\psi}_V(y)\} = \{\widehat{p}_V^{\pm}\}$. Then

$$d_Q(\hat{x}, \widehat{\Psi}(y)) \leqslant (\chi' + |\mathcal{V}|)(L\kappa + L),$$

so to conclude, it is sufficient to prove:

Claim 1. Let $\mathcal{V}' \subseteq \mathcal{V}$ be a set of pairwise transverse domains with maximal cardinality. Then $|\mathcal{V}'| \leqslant P$, where P depends only on the HHS parameters. Hence there exists $P_1 \in \mathbb{N}$, depending only on the HHS parameters, such that $|\mathcal{V}| \leqslant P_1$.

Proof of Claim 1. Since $r > L\kappa + L$, each $V \in \mathcal{V}$ is $\sqsubseteq_{\mathfrak{U}}$-minimal, so $T_V = T_V^e$ and $q_V : T_V \to \widehat{T}_V$ is the identity.

Since both $\widehat{x}$ and $\widehat{\Psi}(y)$ are consistent, one of the following holds:

(1) for all $V \in \mathcal{V}'$, we have $\widehat{x}_V = \widehat{p_V^-}$ and $\widehat{\psi}_V(y) = \widehat{p}_V^{+}$;
(2) for all $V \in \mathcal{V}'$ we have $\widehat{x}_V = \widehat{p}_V^{+}$ and $\widehat{\psi}_V(y) = \widehat{p_V^-}$.

We assume that (1) holds; the other case is similar.

Test points and passing up. Since $\mathcal{V}'$ consists of transverse elements and $d_V(\pi_V(p^+), \pi_V(p^-)) > K/L - L \geqslant 50E$ for all $V \in \mathcal{V}'$, we have a total order $\prec$ on $\mathcal{V}'$ given by $V \prec W$ if $d_V(\rho_V^W, \pi_V(p^+)) \leqslant E$ (see [BHS19, Sec. 2]). Since $\hat{x}, \widehat{\Psi}(y) \in Q$, Lemma 3.19 implies $|\mathcal{V}'| < \infty$, and we may assume $\mathcal{V}' \neq \varnothing$. Hence $\mathcal{V}'$ contains a unique $\prec$–minimal element V_1 and a unique $\prec$–maximal element V_2. Let $\mathcal{V}'' = \mathcal{V}' - \{V_1, V_2\}$, and, for $i \in \{1,2\}$, let $z_i \in P_{V_i}$. (Recall that P_{V_i} is the standard product region from Section 2.9.)

Then for all $V \in \mathcal{V}''$, we have $d_V(\rho_V^{V_1}, z_1) \leqslant E$, and $d_V(\rho_V^{V_2}, z_2) \leqslant E$. Now, since $V_1 \prec V \prec V_2$, we also have $d_V(\rho_V^{V_1}, \pi_V(p^-)) \leqslant E$ and $d_V(\rho_V^{V_2}, \pi_V(p^+)) \leqslant E$, so $d_V(z_1, \pi_V(p^-)) \leqslant 2E$ and $d_V(z_2, \pi_V(p^+)) \leqslant 2E$. Hence $d_V(z_1, z_2) \geqslant K/L - L - 4E \geqslant 50E$.

Let P be the constant from [Dur23, Proposition 4.3], applied to the pair $(K/L - L - 4E, L(L(\kappa+2) + 2\alpha_1) + KL + L)$. This proposition then implies that, if $|\mathcal{V}''| \geqslant P$, then there exists $W \in \mathfrak{S}$ and $\mathcal{V}''' \subseteq \mathcal{V}''$ such that

- $d_W(z_1, z_2) \geqslant KL + L$, so $d_{T_W}(\psi_W(z_1), \psi_W(z_2)) \geqslant K$ and hence $W \in \mathfrak{U}$.
- $V \sqsubsetneq W$ for all $V \in \mathcal{V}'''$.

- $\mathcal{V}'''$ satisfies
$$\mathsf{diam}_W\Big(\bigcup_{V\in\mathcal{V}'''}\rho_W^V\Big) > L(L(\kappa+2)+2\alpha_1).$$

It follows that $\mathsf{diam}_{T_W}(\cup_{V\in\mathcal{V}'}\delta_W^V) > L(\kappa+1)+2\alpha_1$. Let $\mathcal{V}''' = \{V_1,\dots,V_k\}$ be numbered so that $i \leqslant j$ if and only if $\delta_{V_j}^{V_i} = p_{V_j}^-$. Since $L(L(\kappa+2)+2\alpha_1) > L(E'+L)$, we have $k \geqslant 2$ by Lemma 3.10.(2). Then, since x and $\Psi(y)$ are α_1-consistent, we have $x_W \in \mathrm{hull}_{T_W}(p_W^- \cup \mathcal{N}_{\alpha_1}(\delta_W^{V_1}))$ and $\psi_W(y) \in \mathrm{hull}_{T_W}(\mathcal{N}_{\alpha_1}(\delta_W^{V_k}) \cup p_W^+)$. But then $d_{T_W}(x_W,\psi_W(y)) > L\kappa + L$, which is a contradiction.

Bounding $|\mathcal{V}|$. We have shown that $|\mathcal{V}'| \leqslant P+2$, which depends only on the HHS parameters, so, by Ramsey's theorem, we can conclude by taking $P_1 = \mathrm{Ram}(\chi, P+2)$, where $\mathrm{Ram}(\cdot,\cdot)$ denotes the Ramsey number, because no two elements of $\mathcal{V}$ are $\subsetneq$–related, pairwise orthogonal sets have cardinality at most χ, and pairwise transverse subsets have cardinality at most $P+2$. ■

Claim 1 and the inequality preceding it imply
$$d_Q(\widehat{x},\widehat{\Psi}(y)) \leqslant (\chi'+P_1)L(\kappa+1),$$
which completes the proof. □

3.7. **Quasi-isometry.** Given $B, C \geqslant 0$, let $\{\!\{B\}\!\}_C := 0$ if $B < C$ and B otherwise. Given $x \in X$, denote $\widehat{x} := \widehat{\Psi}(x)$, $x_U := \psi_U(x) \in T_U$ and $\widehat{x}_U := \widehat{\psi}_U(x) \in \widehat{T}_U$.

Proposition 3.28. *There exists a constant L' depending only on the HHS parameters such that $\widehat{\Psi}$ is an (L',L')-quasi-isometry.*

Proof. By Proposition 3.26, $\widehat{\Psi}$ is coarsely surjective, with constant depending only on the HHS parameters, so it remains to show that $\widehat{\Psi}$ is a quasi-isometric embedding.

Let $x,y \in X$. For each $U \in \mathfrak{U}$, the map f_U is an (L,L)-quasi-isometry and q_U is 1-lipschitz, so
$$\frac{1}{L}d_{\widehat{T}_U}(\widehat{x}_U,\widehat{y}_U) - L \leqslant \frac{1}{L}d_{T_U}(x_U,y_U) - L \leqslant d_U(x,y) \leqslant L d_{T_U}(x_U,y_U) + L.$$
By Theorem A.1, there exist $S \geqslant L(K+50E)+L$ and $\tau \geqslant 1$ depending only on L and the HHS parameters such that
$$\frac{1}{\tau}\sum_{U\in\mathfrak{U}}\{\!\{d_{T_U}(x_U,y_U)\}\!\}_S - \tau \leqslant d_X(x,y) \leqslant \tau\sum_{U\in\mathfrak{U}}\{\!\{d_{T_U}(x_U,y_U)\}\!\}_S + \tau. \tag{1}$$

To bound the left-hand side of (1) from below in terms of $d_Q(\widehat{x},\widehat{y})$, we need to control the contribution to $d_Q(\widehat{x},\widehat{y})$ from the set $\mathcal{V} := \{U \in \mathfrak{U} : d_{T_U}(x_U,y_U) < S \text{ and } \widehat{x}_U \neq \widehat{y}_U\}$. By Lemma 3.19.(2), there are at most $2\chi'$ domains $U \in \mathcal{V}$ such that $\{\widehat{x}_U,\widehat{y}_U\} \nsubseteq \{\widehat{p_U^-},\widehat{p_U^+}\}$, so these contribute at most $2\chi' S$ to $d_Q(x,y)$. For all remaining $U \in \mathcal{V}$, we have $d_U(x,y) \geqslant K/L - L > 50E$ (using that $r > K$ for the case where U is not $\sqsubseteq_{\mathfrak{U}}$-minimal). It follows from [Dur23, Proposition 4.14][4] that there exist constants $P_1, P_2 \in \mathbb{N}$, depending only on $LS+L$ and the HHS parameters, such that:

- for all but P_1 elements $V \in \mathcal{V}$, there exists $W \in \mathfrak{U}$ such that $d_W(x,y) \geqslant LS+L$ and $V \sqsubseteq W$;
- for any $W \in \mathrm{Rel}_{LS+L}(x,y)$, if $\mathcal{W} := \{V \in \mathcal{V} : V \subsetneq W\}$, then $|\mathcal{W}| \leqslant P_2 d_W(x,y) + P_2$.

[4]The proposition in [Dur23] is stated for rays, but the proof works for points in X. Indeed, it uses Lemmas 4.4 and 4.10 in *loc. cit.*, which are written for interior points as well as rays, and the rest of proof of Proposition 4.14 does not depend on any of the sets involved being unbounded.

Therefore

$$d_Q(\widehat{x},\widehat{y}) \leqslant 2\chi' S + P_1 S + (4SP_2L+1)\sum_{U\in\mathfrak{U}} \{\!\{d_{T_U}(x_U,y_U)\}\!\}_S\,.$$

Combined with (1), we have shown that $d_Q(\widehat{x},\widehat{y}) \leqslant \tau_1 d_X(x,y)+\tau_1$, where $\tau_1 \geqslant 1$ depends only on the HHS parameters.

It remains to bound the right-hand side of (1) from above in terms of $d_Q(\widehat{x},\widehat{y})$. Let $U\in\mathfrak{U}$ and suppose that $d_{T_U}(x_U,y_U) > d_{\widehat{T}_U}(\widehat{x}_U,\widehat{y}_U)$. Let $\{C_1,\dots,C_k\}$ be the clusters in T_U whose intersections with the geodesic $[x_U,y_U]_{T_U}\subseteq T_U$ from x_U to y_U have strictly positive diameter. For each i, let $C_i' := (C_i\cap[x_U,y_U]_{T_U}) - (\mathcal{N}_{E'}(x_U)\cup\mathcal{N}_{E'}(y_U))$ and let $\mathcal{V}_i := \{V\in\mathfrak{U} : V\subsetneq U \text{ and } \delta_U^V\subseteq C_i\}$. The set $\{\delta_U^V : V\in\mathcal{V}_i\}$ is $(R+r)$–dense in C_i and it follows that there is a subset $\mathcal{V}_i'\subseteq\mathcal{V}$ such that:

- each $V\in\mathcal{V}_i'$ is $\sqsubseteq_{\mathfrak{U}}$-minimal;
- $\delta_U^V\subseteq C_i'$ for each $V\in\mathcal{V}_i'$;
- $\{\delta_U^V : V\in\mathcal{V}_i'\}$ is $2(R+r)$-dense in C_i'.

We then have $|\mathcal{V}_i'| \geqslant \mathsf{diam}(C_i')/4(R+r)-1$, using that $\mathsf{diam}(\delta_U^V)\leqslant E' < R$ for each $V\subsetneq U$. For each $V\in\mathcal{V}_i'$, the fact that $\delta_U^V\subseteq C_i'$ implies that $x_V=p_V^-$ and $y_V=p_V^+$, or vice versa, so, by Remark 3.13, $d_{\widehat{T}_V}(\widehat{x}_V,\widehat{y}_V)=d_{T_V}(x_V,y_V)\geqslant K$. Thus we have:

$$\begin{aligned}
d_{T_U}(x_U,y_U) &= d_{\widehat{T}_U}(\widehat{x}_U,\widehat{y}_U) + \sum_{i=1}^k \mathsf{diam}(C_i\cap[x_U,y_U]_{T_U})\\
&\leqslant d_{\widehat{T}_U}(\widehat{x}_U,\widehat{y}_U) + 4(R+r)\sum_{i=1}^k(|\mathcal{V}_i'|+1) + 2kE'\\
&\leqslant (1+2E')d_{\widehat{T}_U}(\widehat{x}_U,\widehat{y}_U) + 4(R+r)\sum_{i=1}^k K|\mathcal{V}_i'| + 4(R+r)\\
&\leqslant (1+2E')d_{\widehat{T}_U}(\widehat{x}_U,\widehat{y}_U) + 4(R+r)\sum_{i=1}^k\sum_{V\in\mathcal{V}_i} d_{\widehat{T}_V}(\widehat{x}_V,\widehat{y}_V) + 4(R+r).
\end{aligned}$$

Now, [DMS23, Lemma 2.15] provides a constant N, depending only on the HHS parameters, such that each $V\in\mathfrak{U}$ is properly nested in at most N elements of $\mathfrak{U}$. Therefore:

$$\sum_{U\in\mathfrak{U}} d_{T_U}(x_U,y_U) \leqslant (1+2E'+4N(R+r))d_Q(\widehat{x},\widehat{y}) + 4(R+r).$$

This completes the proof. □

3.8. **Quasi-median.**

Proposition 3.29. *The map $\widehat{\Psi}$ is $\chi'(L(C_0+C_1)+2L)$-quasimedian.*

Proof. Let $x,y,z\in X$. We keep the same convention as in the previous section. Let $m\in\mu(x,y,z)$, let $\ddot{m}=(\ddot{m}_U)_{U\in\mathfrak{U}}\in Q$ be the median of $\widehat{x},\widehat{y},\widehat{z}$, and let $a,b\in\{x,y,z\}$ be distinct points. For any $U\in\mathfrak{U}$ and any geodesic γ in $\mathcal{C}U$ joining $\pi_U(a)$ and $\pi_U(b)$, we have $d_U(\pi_U(m),\gamma)\leqslant C_0+C_1$. It follows that $d_{T_U}(m_U,[a_U,b_U]_{T_U})\leqslant L(C_0+C_1)+2L$, so $d_{\widehat{T}_U}(\widehat{m}_U,[\widehat{a}_U,\widehat{b}_U]_{\widehat{T}_U})\leqslant L(C_0+C_1)+2L$. By definition, $\ddot{m}_U$ lies on $[\widehat{a}_U,\widehat{b}_U]_{\widehat{T}_U}$ for all distinct $a,b\in\{x,y,z\}$, so this implies that $d_{\widehat{T}_U}(\widehat{m}_U,\ddot{m}_U)\leqslant L(C_0+C_1)+2L$ for all $U\in\mathfrak{U}$. Then, since $r,K > L(C_0+C_1)+2L$, we have $\{\widehat{m}_U,\ddot{m}_U\}\neq\{\widehat{p}_U^{\pm}\}$ for all $U\in\mathrm{Rel}_Q(\widehat{m},\ddot{m})$. By Lemma 3.19.(1) we then have $|\mathrm{Rel}_Q(\widehat{m},\ddot{m})|\leqslant\chi'$, so $d_Q(\widehat{m},\ddot{m})\leqslant\chi'(L(C_0+C_1)+2L)$. □

3.9. **Cocompact quasilines.** We conclude the section with:

Proof of Proposition 3.2. Fix a point $z_0 \in X$. Let $\mathrm{supp}(p^\pm) = \{U_1, \ldots, U_r\}$, where $r \leqslant \chi$ since $\mathrm{supp}(p^\pm)$ is a set of pairwise orthogonal domains. Since $X = H_{M,z_0}(p^-, p^+)$, the points $\pi_{U_i}(p^-), \pi_{U_i}(p^+)$ are distinct points of $\partial \mathcal{C}U_i$ for each i, and $\pi_{U_i}(X)$ is M–close to some $(1, 20E)$–quasigeodesic joining $\pi_{U_i}(p^\pm)$. Hence $\mathcal{C}U_i$ is a uniform quasiline, for $1 \leqslant i \leqslant r$. Moreover, Proposition 3.6 provides a finite-index subgroup $G' \leqslant G$ fixing p^-, p^+ and (passing if needed to a further finite-index subgroup) $G' \cdot U_i = U_i$ and G' therefore acts on $\mathcal{C}U_i$ by isometries, fixing the two-point set $\partial \mathcal{C}U_i$ pointwise. Since $\pi_{U_i}(G \cdot z_0)$ is unbounded by hypothesis, so is $\pi_{U_i}(G' \cdot z_0) = G' \cdot \pi_{U_i}(z_0)$; therefore G' contains elements acting on $\mathcal{C}U_i$ loxodromically, so the action of G' on $\mathcal{C}U_i$ is cobounded. Proposition 3.6 provides a G'–action on a line T_{U_i} and an equivariant quasi-isometry $f_i : \mathcal{C}U_i \to T_{U_i}$, for each i.

For each i, let $\mathfrak{U}_i = \{V \in \mathfrak{U} : V \sqsubseteq U_i\}$. Then $V \perp V'$ whenever $V \in \mathfrak{U}_i$ and $V' \in \mathfrak{U}_j$ with $i \neq j$. Let $\mathfrak{S}_i = \mathfrak{S} - \bigcup_{j \neq i} \mathfrak{U}_j$, so that we have a canonical cone-off $c_i : (X, \mathfrak{S}) \to (\check{X}_i, \mathfrak{S}_i)$. Applying Theorem 3.1 to each G'–action on $(\check{X}_i, \mathfrak{S}_i)$, we obtain a median space Q_i and a G'–equivariant quasi-isometry $\Psi_i : \check{X}_i \to Q_i$ such that the diagram:

$$\begin{tikzcd} X \arrow[r, "c_i"] \arrow[d, "\Psi"'] & \check{X}_i \arrow[d, "\Psi_i"] \\ Q \arrow[r, "\bar{c}_i"'] & Q_i \end{tikzcd}$$

commutes, where $\bar{c}_i$ is a G'–equivariant 1–lipschitz median-preserving map. Indeed, Q_i is obtained from $Q \subseteq \widehat{\mathcal{Y}}$ by discarding the V–coordinates from consistent tuples for all $V \notin \mathfrak{U}_i$. In other words, $\bar{c}_i$ is the restriction of the natural projection $\prod_{U \in \mathfrak{U}} \widehat{T}_U \to \prod_{U \in \mathfrak{U}_i} \widehat{T}_U$. From this, one easily checks that $\prod_i \bar{c}_i : Q \to \prod_i Q_i$ is an isometry.

Moreover, since $\bar{c}_i$ is the restriction to Q of the natural projection $\prod_{V \in \mathfrak{U}} \widehat{T}_V \to \prod_{U \in \mathfrak{U}_i} \widehat{T}_U$, and the decomposition $\mathfrak{U} = \bigsqcup_i \mathfrak{U}_i$ is preserved by G, there is an isometric action of G on $\prod_i Q_i$ that extends the diagonal action of G' and makes $\prod_i \bar{c}_i$ a G–equivariant map.

Since Theorem 3.1 implies that the Q_i are proper spaces, G'–cocompactness follows once we prove that G' acts on Q_i coboundedly. This, plus the fact that Q_i is a quasiline, will follow once we prove that $\pi_{U_i} : \check{X}_i \to \mathcal{C}U_i$ is a quasi-isometry, since we already saw that the G'–action on $\mathcal{C}U_i$ is cobounded and the latter space is a quasiline. By the distance formula for $(\check{X}_i, \mathfrak{S}_i)$, the fact that π_{U_i} is a quasi-isometry will follow once we have proved the "moreover" clause of the proposition.

So, to conclude, it suffices to exhibit B such that $\mathsf{diam}(\mathcal{C}V) < B$ unless $V = U_i$ for some i. If $V \pitchfork U_i$ or $U_i \subsetneq V$ for some i, then consistency uniformly bounds $\mathsf{diam}(\mathcal{C}V)$, and if $V \perp U_i$ for all i, then Lemma 3.4 gives the desired bound. Hence it remains to consider the case of $V \subsetneq U_i$. For any such V, we have $\mathsf{diam}(\mathcal{C}V) < \infty$ by Proposition 3.6, so it suffices to show that $\mathfrak{U}_i$ contains finitely many G'–orbits.

Let $\Omega \subseteq \mathcal{C}U_i$ be a bounded set such that $G' \cdot \Omega = \mathcal{C}U_i$ and $\mathcal{C}U_i - \Omega$ has two unbounded components Ω^-, Ω^+. Choose $z^-, z^+ \in X$ such that $\pi_{U_i}(z^\pm) \subseteq \Omega^\pm$ and $d_{U_i}(z^\pm, \Omega) > 100E$. Recall that $\mathsf{diam}(\mathcal{C}V) \geqslant K$ for all $V \in \mathfrak{U}_i$. Moreover, by consistency and bounded geodesic image, if $V \in \mathfrak{U}_i$ and $\rho^V_{U_i} \cap \Omega \neq \varnothing$, then (up to relabelling) $d_V(z^\pm, \pi_V(p^\pm)) \leqslant E$, so $d_V(z^-, z^+) > K - 2E > E$. Hence [CRHK24, Lem. 13.5] implies that $|\{V \in \mathfrak{U}_i : \rho^V_{U_i} \cap \Omega \neq \varnothing\}| < \infty$. But if $V \in \mathfrak{U}_i$, then $G' \cdot V$ contains some gV in the above set, so we are done. $\square$

4. HHS actions with bounded orbits

The purpose of this section is to prove:

Proposition 4.1. *Let $(X, \mathfrak{S})$ be an HHS. Then there exists k_0, depending only on the HHS parameters, such that for all $k_1 \geqslant k_0(2k_0+1)$, there exist $k_2, \dots, k_7$, depending only on k_1 and the HHS parameters, and T' depending only on $k_1, \dots, k_4$ and the HHS parameters, such that the following holds.*

Let $G \leqslant \mathrm{Isom}(X)$ act on $(X, \mathfrak{S})$ by HHS automorphisms, with bounded orbits in X. Let $Z = Z(G, X, k_1) := \{x \in X : d(x, gx) \leqslant k_1 \ \forall g \in G\}$. Then

(1) $Z \neq \varnothing$;
(2) there exists a non-empty injective space Y, equipped with the trivial G action, and a G–equivariant (k_2, k_2)–quasi-isometry $q : Z \to Y$, where Z has the subspace metric inherited from X;
(3) Z is (k_3, k_4)–EHQC in $(X, \mathfrak{S})$ and a k_5-coarse median subalgebra;
(4) Z is a k_6–coarse lipschitz retract of X;
(5) there is an unbounded, increasing function $\omega_0 : [0, \infty) \to [0, \infty)$, depending only on the HHS parameters, such that $\mathsf{diam}_X(G \cdot x) \geqslant \omega_0(d(x, Z))$ for all $x \in X$;
(6) if $G_0 \trianglelefteq G$ has the property that $gV \not\perp V$ for all $V \in \mathfrak{S}$ such that $\pi_V(Z)$ has diameter at least T', then $H_\theta(Z) \subseteq Z(G_0, X, k_7)$. This holds in particular if G_0 is contained in the intersection of all subgroups of G with index $\leqslant \chi!$.

Finally, suppose that there exists B such that $\mathsf{diam}(\mathcal{C}U) \leqslant B$ or $\mathsf{diam}(\mathcal{C}U) = \infty$ for all $U \in \mathfrak{S}$. Then one can replace χ in item (6) with the maximal r such that $\mathfrak{S}$ contains a pairwise-orthogonal set $\{U_1, \dots, U_r\}$ with each $\mathcal{C}U_i$ unbounded.

Proof. In view of Lemma 2.47, Theorem A.5 and Corollary 2.45, we can and shall assume that X has been chosen in its G–equivariant quasi-isometry class so that there is a constant k_0, depending only on the HHS parameters, and a G–equivariant, k_0–coarsely surjective, (k_0, k_0)–quasi-isometry $\iota : X \to \mathrm{Env}(X)$, where $\mathrm{Env}(X)$ is the injective hull of X.

Construction of Y. Since G acts on $\mathrm{Env}(X)$ with bounded orbits, the set $Y \subseteq \mathrm{Env}(X)$ of points fixed by G is nonempty and injective by [Lan13, Prop. 1.2]. Hence $Z_0 := \iota^{-1}(\mathcal{N}_{k_0}(Y))$ is a nonempty, G–invariant subset of X. The same computation as in the proof of Corollary 2.46 shows that $d(x, gx) \leqslant k_0(2k_0+1)$ for all $x \in Z_0$.

Claim 1. There is an unbounded, increasing function $t : [0, \infty) \to [0, \infty)$, depending only on the HHS parameters, such that $\mathsf{diam}_X(G \cdot x) \geqslant t(d(x, Z_0))$ for all $x \in X$.

Proof of Claim 1. Let $x \in \mathrm{Env}(X)$ and let $R = \mathsf{diam}_{\mathrm{Env}(X)}(G \cdot x)$. Consider $G \cdot x$ with the metric inherited from $\mathrm{Env}(X)$, and the canonical isometric embedding $\iota : G \cdot x \to \mathrm{Env}(G \cdot x)$ (see [Lan13, Sec. 3]). Note that $\mathsf{diam}(\mathrm{Env}(G \cdot x)) \leqslant R$. On the other hand, by [Lan13, Prop. 3.5], the inclusion $G \cdot x \to \mathrm{Env}(X)$ extends to an embedding $h : \mathrm{Env}(G \cdot x) \to \mathrm{Env}(\mathrm{Env}(X)) = \mathrm{Env}(X)$, and h is G–equivariant. In other words, $\mathrm{Env}(X)$ contains a G–invariant injective subspace $\mathrm{Env}(G \cdot x)$ that has diameter R and contains $G \cdot x$. Proposition 1.2 of [Lan13], applied to this subspace, shows that $d_{\mathrm{Env}(X)}(Y, G \cdot x) \leqslant R$, so $d_{\mathrm{Env}(X)}(Y, x) \leqslant 2R$. In particular, if $z \in X$, then $d_{\mathrm{Env}(X)}(\iota(z), Y) \leqslant 2\,\mathsf{diam}_{\mathrm{Env}(X)}(G \cdot \iota(z))$, which implies the claim since ι is an equivariant, k_0–coarsely surjective, (k_0, k_0)–quasi-isometry. ■

Let $k_1 \geqslant k_0(2k_0+1)$ be given, and let $Z = Z(G, X, k_1)$ be as in the statement.

Claim 2. The set Z is nonempty and G–invariant, and the Hausdorff distance in X between Z and Z_0 is bounded above in terms of k_0 and k_1.

Proof of Claim 2. That Z is G–invariant is immediate from the definition. Since $z \in Z_0$ implies $\mathsf{diam}(G \cdot z) \leqslant k_0(2k_0+1)$, we have $Z_0 \subseteq Z$; in particular, $Z \neq \varnothing$. Let t be as in Claim 1. Then $Z \subseteq \mathcal{N}_{t^{-1}(k_1)}(Z_0)$, yielding the required Hausdorff distance bound. ■

Item (5) follows immediately from Claims 1 and 2. Since Y is injective — and therefore an absolute 1-lipschitz retract (see e.g. [Lan13, Prop. 2.2]) — and an isometrically embedded subspace of $\mathrm{Env}(X)$, there is a 1-lipschitz retraction $\mathrm{Env}(X) \to Y$. Using Claim 2 and the fact that ι is (k_0, k_0)–quasi-isometry, we can therefore compose uniformly coarsely lipschitz maps $X \to \mathrm{Env}(X) \to Y \to Z_0 \to Z$ to produce a coarsely lipschitz map $\rho : X \to Z$. Up to uniformly perturbing ρ (here, uniformity allows dependence on k_0, k_1, and the HHS parameters, as in the statement), we can assume it is the identity on Z, proving Item (4).

Construction of q. By Claim 2, $\iota : Z \to \mathrm{Env}(X)$ is a (k_0, k_0)–quasi-isometric embedding whose image is G–invariant and at Hausdorff distance at most $k_0(t^{-1}(k_1)+2)$ from Y. Recall that Y, with the subspace metric inherited from $\mathrm{Env}(X)$, is injective. Let $\{z_i\}_i$ be a subset of Z containing one point in each G–orbit. For each i, let $q(z_i)$ be some point in Y that is $k_0(t^{-1}(k_1) + 2)$–close to $\iota(z_i)$. Given $z \in Z$, let z_i be the element in our set of orbit representatives belonging to $G \cdot z$. Let $q(z) = q(z_i)$. Note that for any $g \in G$ such that $gz_i = z$, we have $q(gz_i) = q(z_i) = gq(z_i)$ since G acts on Y trivially. Hence $q : Z \to Y$ is G–equivariant. Moreover, for any $z = gz_i$, since ι is also equivariant, we have

$$d_{\mathrm{Env}(X)}(q(z), \iota(z)) = d_{\mathrm{Env}(X)}(q(z_i), g\iota(z_i)) = d_{\mathrm{Env}(X)}(q(z_i), \iota(z_i)) \leqslant k_0(t^{-1}(k_1) + 2),$$

so q is at uniformly bounded distance from the restriction of ι and is therefore a uniform quasi-isometry.

Proof of (3). We first check that Z is a coarse median subalgebra (Definition 2.21) with constant depending only on the HHS parameters and k_1. First recall that by Condition (C1) of the definition of a coarse median (Definition 2.16), there exists α, depending only on the coarse median parameters (and hence, by Corollary A.3, only on the HHS parameters) such that the coarse median $\mu : X^3 \to X$ is (α, α)–coarsely lipschitz. Now let $z_1, z_2, z_3 \in Z$, let $m := \mu(z_1, z_2, z_3)$, and let $g \in G$. Then $d(z_i, gz_i) \leqslant k_1$ for $i \in \{1, 2, 3\}$, by the definition of Z. Hence $d(m, gm) \leqslant 3\alpha k_1 + \alpha$, which depends only on k_1 and the HHS parameters. Therefore Claims 1 and 2 imply that $d(m, Z)$ is bounded in terms of k_1 and the HHS parameters.

Let us now check that Z is uniformly EHQC. To this end, let $p^+, p^- \in Z$. We will use Theorem 3.1 to find to the required hierarchy path.

Let $W := G \cdot H_\theta(p^+, p^-)$ and let $M := \theta + Ek_1 + 3E$. Then $H_\theta(p^+, p^-) \subseteq W \subseteq H_M(p^+, p^-)$. It follows from Lemma 2.36 that W is κ'-hierarchically quasiconvex, where $\kappa'(0) := h_M(0)$ and $\kappa'(r) := h_\theta(r)$ if $r > 0$. Let $\mathfrak{S}_W$ be the HHS structure on W from Lemma 2.28 and recall from that lemma that the action of G on $(W, \mathfrak{S}_W)$ is by HHS automorphisms. The hierarchical hull $H^W_M(p^+, p^-)$ of p^+, p^- in W with respect to the constant M is just the intersection $W \cap H^X_M(p^+, p^-)$, so $W = H^W_{M,z_0}(p^+, p^-)$ for all $z_0 \in W$. Thus, by Theorem 3.1, there is a complete connected median space (Q, d_Q), of rank at most χ, equipped with a G–action and a G-equivariant L-quasi-isometry $\Psi : W \to Q$, where $L \geqslant 1$ only depends on M and HHS parameters for $(W, \mathfrak{S}_W)$, and hence only on the HHS parameters for $(X, \mathfrak{S})$.

By [Bow16, Thm. 1.1], Q has a metric σ such that (Q, σ) is a CAT(0) space, any isometry of (Q, d_Q) is an isometry of (Q, σ), and $d_Q/\sqrt{\chi} \leqslant \sigma \leqslant d_Q$. Let $\gamma_Q \subseteq Q$ be the unique σ–geodesic connecting $\Psi(p^+)$ and $\Psi(p^-)$. Then $\gamma_W := \Psi^{-1}(\gamma_Q)$ is a (C, C)-quasigeodesic, where the constant C depends only on L and χ. The σ–geodesic γ_Q is also a (reparametrised) d_Q-geodesic (see the proof of [Bow16, Thm. 8.3]). Since Ψ is quasimedian, it follows from Proposition 2.25 that γ_W is a k_3–hierarchy path in $(W, \mathfrak{S}_W)$ and therefore in $(X, \mathfrak{S})$, where k_3 depends only on the HHS parameters of X. Given $x \in \gamma$, convexity of the metric σ implies that $\sigma(x, g\gamma) \leqslant \max\{\sigma(g_1\Psi(p^+), g_2\Psi(p^+)), \sigma(g_1\Psi(p^-), g_2\Psi(p^-))\}$ for all $g \in G$. Therefore $\mathsf{diam}_\sigma(G \cdot x) \leqslant Lk_1 + L$ for all $x \in \gamma$ and $\mathsf{diam}_X(G \cdot y) \leqslant L^2(k_1 + 1) + L$ for all $y \in \gamma_W$. By Claim 1 and Claim 2, we then have $\gamma_W \subseteq \mathcal{N}_{k_4}(Z)$ for some $k_4 \geqslant 0$ depending only on the HHS parameters.

Proof of (6)**.** Let $T := Ek_1 + 4E$, let $T' := k_3(2T + k_3) + k_4$ and let $\mathfrak{U} := \bigcup_{a,b \in Z} \mathrm{Rel}_{T'}(a, b)$. Since G preserves Z, the set $\mathfrak{U}$ is G–invariant.

Fix $V \in \mathfrak{U}$ and $g \in G$. Note that gV and V cannot be $\subsetneq$–related. We next argue that gV and V cannot be transverse. Indeed, suppose $gV \pitchfork V$. It follows from the definition of $\mathfrak{U}$ and the fact that Z is (k_3, k_4)-EHQC that there exists $a \in Z$ such that $d_V(a, \rho_V^{gV}) > T$ and $d_V(a, \rho_V^{g^{-1}V}) = d_{gV}(ga, \rho_{gV}^V) > Ek_1 + 4E$. However the consistency axiom [BHS19, Def. 1.1.(4)] implies that $d_{gV}(a, \rho_{gV}^V) \leqslant E$, so $d_{gV}(a, ga) \geqslant Ek_1 + 3E$, so $d(a, ga) \geqslant k_1 + 2E$, which contradicts the assumption that $a \in Z$. Therefore either $gV = V$ or $gV \perp V$.

Now assume that $G_0 \trianglelefteq G$ has the property that $gV \not\perp V$ for all $V \in \mathfrak{U}$ and $g \in G_0$. The preceding discussion implies that G_0 acts on $\mathfrak{U}$ trivially. Let $x \in H_\theta(Z)$ and $g \in G_0$. A standard hyperbolic geometry argument shows that $d_U(x, gx) \leqslant k_7'$ for all $U \in \mathfrak{U}$, where k_7' is a constant which depends only on E and k_1. If $U \in \mathfrak{S} - \mathfrak{U}$, then $\mathrm{hull}_U(Z)$ is uniformly bounded in terms of E and T', so $\pi_U(H_\theta(Z))$ is uniformly bounded in terms of E and T'. It follows by the distance formula (Theorem A.1) that $d(x, gx) \leqslant k_7$, where k_7 depends only on k_1 and the HHS parameters. This proves the first part of item (6).

To conclude, suppose that G_0 is contained in the intersection of all subgroups of G with index $\leqslant \chi!$. We need to show that $G_0 \cdot V = \{V\}$ for each $V \in \mathfrak{U}$. Now, we have shown that $G \cdot V$ is a pairwise orthogonal set, and therefore $|G \cdot V| \leqslant \chi$. Hence $\mathrm{Stab}_G(V)$ has index at most $\chi!$ in G. Therefore, $G_0 \leqslant \mathrm{Stab}_G(V)$ for all $V \in \mathfrak{U}$, as required.

Finally, suppose that there exists $B < \infty$ such that for all $U \in \mathfrak{S}$, either $\mathcal{C}U$ is unbounded, or $\mathsf{diam}(\mathcal{C}U) \leqslant B$. In the preceding argument, we can assume that $T' \geqslant B$, by, for instance, choosing k_1 sufficiently large. Thus $\mathfrak{U}$ consists only of elements U with $\mathcal{C}U$ unbounded, and the above argument worked as long as G_0 is contained in each subgroup of index at most $r!$, where r bounds the size of pairwise orthogonal sets in $\mathfrak{U}$. ☐

5. Some invariant subspaces

The next definition abstracts a property of the action by HHS automorphisms of an HHG on itself (see Definition 2.11) by [DHS20, Thm. 3.1].

Definition 5.1 (Hierarchically semisimple)**.** An action of a group H on an HHS $(X, \mathfrak{S})$ by HHS automorphisms is *hierarchically semisimple* if, for all $g \in H$ and all $U \in \mathfrak{S}$ such that $gU = U$ and $\langle g \rangle$ has unbounded orbits in $\mathcal{C}U$, the isometry g of $\mathcal{C}U$ is loxodromic. ✽

Let $(X, \mathfrak{S})$ be an HHS and let $A \leqslant \mathrm{Isom}(X)$ be an infinite finitely generated virtually abelian group acting hierarchically semisimply on $(X, \mathfrak{S})$ by HHS automorphisms, with the A–orbits in X unbounded. The goal of this section is to analyse some canonical A-invariant HQC and EHQC subspaces, using the results of the previous sections.

By Lemma 2.47 and Lemma 2.49, we can assume that X is a graph and the A–action on X is free. For convenience, up to a uniform enlargement of the HHS parameters, we can and shall assume that $(X, \mathfrak{S})$ is *normalised* in the sense of [BHS19, Rem. 1.3], which is to say that π_U is E–coarsely surjective for all $U \in \mathfrak{S}$. Recall that χ is the complexity of $\mathfrak{S}$.

Lemma 5.2. *There exists a finite index abelian subgroup $A' \leqslant A$ and a pair of points $p^+, p^- \in \partial X$ such that:*

(1) $ap^+ = p^+$ and $ap^- = p^-$ for all $a \in A'$;
(2) $\mathrm{supp}(p^+) = \mathrm{supp}(p^-)$;
(3) for all $U \in \mathrm{supp}(p^\pm)$ we have $\pi_U(p^+) \neq \pi_U(p^-)$;
(4) for any $x \in X$, the projection $\pi_U(A \cdot x)$ is unbounded if and only if $U \in \mathrm{supp}(p^\pm)$.

Proof. Since A is assumed to be finitely generated and to act with unbounded orbits, [PS23, Thm. 5.1] produces $1 \leqslant n \leqslant \chi$ and domains $U_1, \dots, U_n \in \mathfrak{S}$ such that the following hold for any choice of basepoint $x_0 \in X$:

- $U_i \perp U_j$ for $1 \leqslant i < j \leqslant n$ (which is why $n \leqslant \chi$);
- for all $W \in \mathfrak{S}$, if the set $\pi_W(A \cdot x_0)$ is unbounded, then $W \sqsubseteq U_i$ for some $i \leqslant n$;
- $\pi_{U_i}(A \cdot x_0)$ is unbounded for all $i \in \{1, \dots, n\}$;
- A preserves the set $\{U_i\}_{i=1}^n$.

Let $\ddot{A} \leqslant A$ be the kernel of the A–action on $\{U_i\}_{i=1}^n$; note that $[A : \ddot{A}] \leqslant \chi!$. Let $\ddot{A} \leqslant \ddot{A}$ be a finite index abelian subgroup.

For each $i \leqslant n$, the subgroup $\ddot{A}$ acts on $\mathcal{C}U_i$ with unbounded orbits. Let $a_1, \dots, a_k \in \ddot{A}$ generate $\ddot{A}$. If each $\langle a_j \rangle$ acts on $\mathcal{C}U_i$ with bounded orbits, then since $\ddot{A}$ is abelian, $\ddot{A} \cdot \pi_{U_i}(x_0)$, and hence $\pi_{U_i}(A \cdot x_0)$, is bounded, contradicting the choice of U_i. Combined with the hierarchical semisimplicity assumption, this implies that some $b_i \in \ddot{A}$ acts on $\mathcal{C}U_i$ loxodromically, for each $i \in \{1, \dots, n\}$. The classification of actions on hyperbolic spaces (see, for instance, [CCMT15, Prop. 3.1]) therefore allows three possibilities for the $\ddot{A}$–action on $\mathcal{C}U_i$: it can be *lineal, focal,* or *general-type.* Since $\ddot{A}$ is abelian, it contains no rank-2 free semigroup, which rules out general-type and focal actions.

Hence $\partial\mathcal{C}U_i$ contains exactly two points, denoted $p_i^{\pm}$, such that $\{p_i^+, p_i^-\}$ is $\ddot{A}$-invariant; these are the unique fixed points for each loxodromic element of $\ddot{A}$. Let $p^+, p^- \in \partial X$ be the boundary points whose supports are $\operatorname{supp}(p^+) = \operatorname{supp}(p^-) = \{U_1, \dots, U_n\}$, whose constituent boundary points are $p_i^+, p_i^- \in \partial\mathcal{C}U_i$, respectively, for each $i \in \{1, \dots, n\}$, and with coefficients $\{a_U = 1/n : U \in \operatorname{supp}(p^{\pm})\}$. Let $A' \leqslant \ddot{A}$ be a further finite index subgroup which fixes each $p_i^{\pm}$. Then Items (1)–(4) all hold. □

Definition 5.3. Let $P = P(A) \subseteq X$ be the set of x such that $d_V(x, \rho_V^U) \leqslant 10E + \theta$ whenever $U \in \operatorname{supp}(p^{\pm})$ and $V \in \mathfrak{S}$ satisfies $U \subsetneq V$ or $V \pitchfork U$. ✽

Remark 5.4. Note that P is A–invariant, nonempty, and κ'–median quasiconvex, where κ' just depends on the HHS parameters. Thus, by Proposition 2.23 and Lemma 2.28, P is an HHS with parameters depending only on the HHS parameters of X. Up to enlarging E uniformly, we can assume that E is the HHS constant for P. We also normalise (see [BHS19, Rem. 1.3]) so that $\mathsf{diam}(\mathcal{C}U) \leqslant 30E + 2\theta$ for all $U \in \mathfrak{S}$ such that $V \subsetneq U$ or $V \pitchfork U$ for some $V \in \operatorname{supp}(p^+)$. From now on, we work mostly with P and its HHS structure. It may be helpful to observe that in the case where $\operatorname{supp}(p^+)$ has a single element U, the subspace P is defined exactly like the standard product region P_U, except using a different constant; P and P_U are at finite Hausdorff distance. ✽

Definition 5.5. Let $\mathfrak{U} := \{U \in \mathfrak{S} : U \sqsubseteq V \text{ for some } V \in \operatorname{supp}(p^+)\}$ and let $k \geqslant 0$.

- For each $U \in \mathfrak{S} - \operatorname{supp}^{\perp}(p^{\pm})$, let $\gamma_U := \operatorname{hull}_U(\{p^+, p^-\})$ (i.e. the union of all $(1, 20E)$–quasi-geodesics joining $\pi_U(p^-)$ to $\pi_U(p^+)$).
- Let $(\check{P}, \mathfrak{S} - \mathfrak{U})$ be the canonical $\mathfrak{U}$–cone-off space of P from Definition 2.40 and Corollary 2.43 and let $c : P \to \check{P}$ be the (set-theoretic) identity.
- Let $\overline{P}_k = \overline{P}_k(A) := \{x \in P : d_V(x, \gamma_V) \leqslant k \text{ for all } V \in \mathfrak{S} - \operatorname{supp}^{\perp}(p^{\pm})\}$.
- Let $X_k(A) := \{x \in \overline{P}_k : c(x) \in Z(A, \check{P}, k)\}$, where $Z(A, \check{P}, k) = \{x \in \check{P} : d(x, ax) \leqslant k \ \forall a \in A\}$.
- Let k_0 be the constant obtained by applying Proposition 4.1 to the HHS $(\check{P}, \mathfrak{S} - \mathfrak{U})$. Let $k_1 := k_0(2k_0 + 1)$ and let k_2, k_3, k_4 be the corresponding constants from Proposition 4.1. Let $\check{Z} := Z(A, \check{P}, k_1)$. ✽

Proposition 5.6. *There exist constants $k \geqslant k_1$, $M \geqslant M' \geqslant \theta$ and $L_1, L_2, L_3.L_4 \geqslant 1$, depending only on the HHS parameters, such that the following hold.*

(1) $\overline{P} = \overline{P}(A) := \overline{P}_k$ *is nonempty, A–invariant, and uniformly median quasi-convex.*
(2) $X(A) := X_k(A)$ *is a nonempty, A–invariant uniformly coarse median subalgebra.*
(3) *The image $c(X(A))$ is L_1–Hausdorff close to $\breve{Z}$. In addition, there is a (L_1, L_1)–quasi-isometry $\Phi : c(X(A)) \to Y$, where Y is an injective space and, equipping Y with the trivial A–action, Φ is A–equivariant.*
(4) $H_{M',z}(p^-, p^+) \neq \varnothing$ *and* $A \cdot H_{M',z}(p^-, p^+) \subseteq H_{M,z}(p^-, p^+)$ *for all* $z \in X(A)$.
(5) *Fix $z_0 \in X(A)$ and let $H := A \cdot H_{M',z_0}(p^-, p^+)$. Then*
- *H is κ–median convex in P, where κ depends only on the HHS parameters;*
- *there is an A–equivariant, L_2–quasi-median gate map $\mathfrak{g}_H : P \to H$;*
- *there is a connected, proper, finite rank median space Q, equipped with an A–action and an A–equivariant, L_2–quasi-median, (L_2, L_2)–quasi-isometry $\Psi : H \to Q$.*

(6) *The product map $c|_{\overline{P}} \times \mathfrak{g}_H|_{\overline{P}} : \overline{P} \to \breve{P} \times H$ is an A–equivariant, L_3–quasi-median, (L_3, L_3)–quasi-isometry.*
(7) *Let $\xi := \Phi \circ c|_{X(A)}$ and $\eta := \Psi \circ \mathfrak{g}_H|_{X(A)}$. Then $\xi \times \eta : X(A) \to Y \times Q$ is an A–equivariant (L_4, L_4)–quasi-isometry.*
(8) *$X(A)$ is uniformly EHQC and a uniformly coarse lipschitz retract of X.*

Proof. We start with a claim which will help to establish the A–invariance of the various subspaces we are interested in.

Claim 1. If $U \in \mathfrak{S} - \mathrm{supp}^{\perp}(p^{\pm})$ and $a \in A$ then $a\gamma_U = \gamma_{aU}$.

Proof of Claim 1. Let $a \in A$ and $U \in \mathrm{supp}(p^{\pm})$. Recall that there is an element $b \in \ddot{A}$ which acts loxodromically on $\mathcal{C}U$. So aba^{-1} acts loxodromically on $\mathcal{C}aU$ with quasi-axis $a\gamma_U$. Some nonzero power of aba^{-1} belongs to $\ddot{A}$, and therefore has $\{\pi_{aU}(p^+), \pi_{aU}(p^-)\}$ as its unique fixed points. Thus $a\gamma_U = \gamma_{aU}$. It follows that $a\gamma_V = \gamma_{aV}$ for all $V \in \mathfrak{S} - \mathrm{supp}^{\perp}(p^{\pm})$. ■

We now prepare to use the realisation theorem to find k. Let

$$P' := \{x \in P : d_V(x, \gamma_V) \leqslant E \;\forall\; V \in \mathfrak{S} \text{ such that } V \pitchfork W \text{ or } W \sqsubseteq V \text{ for some } W \in \mathrm{supp}(p^{\pm})\}.$$

By the partial realisation axiom ([BHS19, Defn. 1.1.(8)]), $P' \neq \varnothing$.

Recall that $(\breve{P}, \mathfrak{S} - \mathfrak{U})$ is an HHS with parameters $(\breve{E}, \breve{\theta}_u)$ depending only on the HHS parameters of $(P, \mathfrak{S})$, and A acts on $\breve{P}$ by isometries and on $(\breve{P}, \mathfrak{S} - \mathfrak{U})$ by HHS automorphisms. As a set, $\breve{P} = P$, and the identity map $c : (P, d) \to (\breve{P}, \breve{d})$ is A–equivariant and uniformly quasi-median. Since the set $\pi_V(A \cdot y)$ is bounded for all $U \in \mathfrak{S} - \mathfrak{U}$ and $y \in P$, [PS23, Thm. 5.1] implies that A acts on $\breve{P}$ with bounded orbits. Thus Proposition 4.1 implies that $\breve{Z} \neq \varnothing$.

Claim 2. There exists $\kappa > 0$, depending only on the HHS parameters such that the following holds. Let $x_0 \in P'$ and $y_0 \in \breve{P}$. For each $U \in \mathrm{supp}^{\perp}(p^{\pm})$, let $z_U \in \pi_U(y_0)$ and, for each $U \in \mathfrak{S} - \mathrm{supp}^{\perp}(p^{\pm})$, let $z_U \in \gamma_U$ be the closest point projection of $\pi_U(x_0)$ onto γ_U. Then $(z_U)_{U \in \mathfrak{S}}$ is κ-consistent.

Proof of Claim 2. Let $U, V \in \mathfrak{S}$ and suppose $U \pitchfork V$ or $U \sqsubsetneq V$. If $U, V \in \mathrm{supp}^{\perp}(p^{\pm})$ then the required inequality follows from the consistency axioms applied to $y_0 \in \breve{P}$. So suppose without loss of generality that $V \notin \mathrm{supp}^{\perp}(p^{\pm})$.

Suppose that V is not nested in any $W \in \mathrm{supp}(p^{\pm})$. Then the set $\mathcal{W}_V := \{W \in \mathrm{supp}(p^{\pm}) : W \sqsubsetneq V \text{ or } W \pitchfork V\}$ is nonempty and $\pi_V(p^+) = \pi_V(p^-) = \cup_{W \in \mathcal{W}_V} \rho^W_V$. So, since the elements of $\mathcal{W}_V$ are pairwise orthogonal, [DHS17, Lem. 1.5] implies that $\mathsf{diam}(\gamma_V)$ is uniformly bounded in terms of E. Also, $d_V(x_0, z_V) \leqslant E$, by the definition of z_V.

Hence, if $d_V(\rho^U_V, \gamma_V) \leqslant 10E$, then $d_V(\rho^U_V, z_V)$ is bounded uniformly in terms of E, and we are done. Suppose that $d_V(\rho^U_V, \gamma_V) > 10E$, so in particular, $d_V(z_V, \rho^U_V) > 10E$ and

$d_V(x_0,\rho^U_V) > 9E$. If $W \in \mathcal{W}_V$, then $d_V(\rho^U_V,\rho^W_V) > 10E$, so U,W are not $\sqsubseteq$–related (because of [BHS19, Defn. 1.1.(4)]) and are not orthogonal (by [DHS17, Lem. 1.5]), so $U \pitchfork W$. In particular, $W \in \mathcal{W}_U$, which is therefore nonempty, and $\pi_U(p^+) = \pi_U(p^-) = \cup_{W\in\mathcal{W}_U}\rho^W_U$. Thus γ_U has uniformly bounded diameter (in terms of E) and $d_V(x_0,z_U) \leqslant E$.

If $U \pitchfork V$, then $d_U(x_0,\rho^V_U) \leqslant E$, by consistency of x_0. Hence $d_U(\rho^V_U,z_U) \leqslant 2E$, as required. The other possibility is that $U \subsetneq V$. In this case $\mathsf{diam}_U(\rho^V_U(\pi_V(x_0)) \cup \pi_U(x_0)) \leqslant E$, by the consistency axiom for nested domains, applied to x_0 (see [BHS19, Defn. 1.1.(4)]). The bounded geodesic image axiom [BHS19, Defn. 1.1.(7)] further implies that $d_U(\rho^V_U(\pi_V(x_0)),\rho^V_U(z_V)) \leqslant E$, which implies that $d_U(z_U,\rho^V_U(z_V)) \leqslant 3E$.

It remains to consider the case where $V \sqsubseteq W$ for some $W \in \mathrm{supp}(p^\pm)$. Then $U \notin \mathrm{supp}^\perp(p^\pm) \cup \mathrm{supp}(p^\pm)$. If $W' \subsetneq U$ or $W' \pitchfork U$ for some $W' \in \mathrm{supp}(p^\pm)$ then $V \perp W'$ by [BHS19, Defn. 1.1.(3)], so $d_U(\rho^V_U,z_U) \leqslant d_U(\rho^V_U,\rho^{W'}_U) + \mathsf{diam}(\rho^{W'}_U) + d_U(\rho^{W'}_U,z_U) \leqslant 10E$ (using [DHS17, Lem. 1.5] and the definition of z_U). So suppose that $U \subsetneq W'$ for some $W' \in \mathrm{supp}(p^\pm)$. Since $U \not\sqsubseteq V$, we have $W = W'$.

Suppose that $V = W$ and $d_V(z_V,\rho^U_V) > 10E + 2\mathsf{Morse}_E(1,20E)$. Since $x_0 \in P'$, we have $d_V(x_0,z_V) \leqslant E$. Assume without loss of generality that $\pi_V(x_0)$ and z_V are in the same connected component of $\mathcal{C}V - \mathcal{N}_{10E+2\mathsf{Morse}_E(1,20E)}(\rho^U_V)$ as $\pi_V(p^+)$. Then, using [BHS19, Defn. 1.1.(4,7)], we have

$$d_U(x_0,\pi_U(p^+)) \leqslant d_U(x_0,\rho^V_U(\pi_V(x_0)) + d_U(\rho^V_U(\pi_V(x_0)),\pi_U(p^+)) + E \leqslant 3E;$$
$$d_U(x_0,\rho^V_U(z_V)) \leqslant d_U(x_0,\rho^V_U(\pi_V(x_0))) + d_U(\rho^V_U(\pi_V(x_0)),\rho^V_U(z_V)) + E \leqslant 3E.$$

We therefore have $d_U(x_0,z_U) \leqslant 3E$, so $d_U(z_U,\rho^V_U(z_V)) \leqslant 10E$.

Finally, suppose $V \subsetneq W$. There exist uniform neighbourhoods B_U,B_V of the coarse projections of ρ^U_W,ρ^V_W onto the uniform quasiline γ_W such that B_U and B_V each disconnect $\mathcal{N}_\delta(\gamma_W)$, where $\delta \geqslant 0$ is a uniform constant (depending only on E and θ) such that $\mathcal{N}_\delta(\gamma_W)$ is connected. Let $b^-,b^+ \in X$ be such that $\pi_W(b^-),\pi_W(b^+)$ are points in the connected components of $\mathcal{N}_\delta(\gamma_W) - (B_U \cup B_V)$ containing $\pi_W(p^-)$ and $\pi_W(p^+)$ respectively. Using the definition of z_U, hyperbolicity of $\mathcal{C}V$ and the bounded geodesic image axiom, we find that $\pi_U(b^\pm) \in \mathcal{N}_E(\pi_U(p^\pm))$ and $\pi_V(b^\pm) \in \mathcal{N}_E(\pi_V(p^\pm))$. Applying [BHS19, Lem. 2.6] to the triple b^+,b^-,x shows that z_U,z_V satisfy the κ'-consistency condition, where κ' depends only on the HHS parameters. ■

Let r_0 be the constant from the realisation theorem (Theorem 2.15) and let $k := \kappa r_0 + k_1$. Then the realisation theorem together with Claim 2 imply that $\overline{P} := \overline{P}_k$ and $X(A) := X_k(A)$ are nonempty, and $\overline{P}$ is uniformly median quasi-convex. Claim 1 implies that $\overline{P}$ is A–invariant. Thus item (1) holds.

Using Proposition 4.1, there is a constant $k' \geqslant k$ such that :

- $d_{Haus}(c(X(A)),\check{Z}) \leqslant k'$ and $d_{Haus}(c(X(A)),Z(A,\check{P},k)) \leqslant k'$;
- there is a non-empty injective space Y equipped with the trivial A–action and a (k',k')–quasi-isometry $q : Z(A,\check{P},k) \to Y$;
- the sets $\check{Z}$ and $Z(A,\check{P},k)$ are (k',k')-EHQC and k'-coarse median subalgebras of $\check{P}$.

It follows that, for some $L_1 \geqslant k'$ depending only on k' and E, the subset $c(X(A))$ is (L_1,L_1)-EHQC and a L_1-coarse median subalgebra of $\check{P}$, and the restriction Φ of q to $c(X(A))$ is an (L_1,L_1)-quasi-isometry $c(X(A)) \to Y$. For all $U \in \mathfrak{S} - \mathfrak{U}$ the projection $\pi_U(X(A))$ is uniformly quasiconvex in $\mathcal{C}U$, so it follows that $X(A)$ is also a uniformly coarse median subalgebra of P. Thus items (2) and (3) hold.

Almost invariant hulls. Let $M' := k + \theta$ and let $z \in X(A)$. The hull $H_{M',z}(p^-,p^+)$ consists of all points $x \in X$ such that $d_U(x,z) \leqslant M'$ if $U \in \mathrm{supp}^\perp(p^\pm)$ and $d_U(x,\gamma_U) \leqslant M'$ if $U \in \mathfrak{S} - \mathrm{supp}^\perp(p^\pm)$. In particular, $z \in H_{M',z}(p^-,p^+)$, so $H_{M',z}(p^-,p^+) \neq \varnothing$.

Let $x \in H_{M',z}(p^-,p^+)$ and $U \in \mathfrak{S}$. If $U \in \operatorname{supp}^\perp(p^\pm)$ then $a^{-1}U \in \operatorname{supp}^\perp(p^\pm)$ and

$$d_U(ax,z) = d_{a^{-1}U}(x,a^{-1}z) \leqslant d_{a^{-1}U}(x,z) + d_{a^{-1}U}(z,a^{-1}z) \leqslant M' + k.$$

If $U \notin \operatorname{supp}^\perp(p^\pm)$ then $d_U(ax,\gamma_U) = d_{a^{-1}U}(x,\gamma_{a^{-1}U}) \leqslant M'$. Taking $M := M'+k$, this proves that $A \cdot H_{M',z}(p^-,p^+) \subseteq H_{M,z}(p^-,p^+)$. Thus (4) holds.

Fix $z_0 \in X(A)$ and let $H := A \cdot H_{M',z_0}(p^-,p^+)$. Since $H_{M',z_0}(p^-,p^+) \subseteq H \subseteq H_{M,z_0}(p^-,p^+)$, Lemma 2.36 implies that H is uniformly HQC, and Lemma 2.23 implies that H is uniformly median-quasiconvex. Hence it admits a gate map and, since the A–action on X is free, we can assume that the gate map $\mathfrak{g}_H : P \to H$ is A–equivariant by Lemma 2.27. The gate map $\mathfrak{g}_H$ is uniformly coarsely lipschitz and uniformly quasimedian, by Construction 2.26. Recall that there is an HHS structure $(H,\mathfrak{S}_H)$ on H with parameters depending only on the HHS parameters of $(P,\mathfrak{S})$ and the quasiconvexity function of H (see Lemma 2.28 and Remark 2.29), which in turn depends only on the HHS parameters of $(P,\mathfrak{S})$. By Theorem 3.1, there is a complete, connected, finite rank, proper median space Q, equipped with an isometric action of A and a A-equivariant, uniformly quasi-median, uniform quality quasi-isometry $\Psi : H \to Q$. This proves item (5).

Quasi-isometry to a product.

Claim 3. There exists a constant T, depending only on the HHS parameters of $(X,\mathfrak{S})$, such that for all $T' \geqslant T$ there exists L_H, depending only on the HHS parameters of $(P,\mathfrak{S})$ and T', such that

$$\sum_{U \in \mathfrak{U}} \{\!\{d_U(x_1,x_2)\}\!\}_{T'} \leqslant L_H d_H(\mathfrak{g}_H(x_1),\mathfrak{g}_H(x_2)) + L_H$$

for all $x_1,x_2 \in \overline{P}$.

Proof of Claim 3. Since the hierarchical quasiconvexity parameters for H are uniform, there is a uniform constant B_0 such that for all $U \in \mathfrak{U}$ and $x \in P$, the projection $\pi_U(\mathfrak{g}_H(x))$ is B_0–close to the image of $\pi_U(x)$ under the coarse closest-point projection to the uniformly quasiconvex set $\pi_U(H) \subseteq \mathcal{C}U$ (see Construction 2.26).

If $U \in \mathfrak{S}-\mathfrak{U}$, then $\pi_U(H)$ has diameter bounded by a uniform constant T_0. Indeed, if $U \pitchfork U_i$ or $U_i \subsetneq U$ for some U_i, then $\mathsf{diam}(\mathcal{C}U) \leqslant 30E + 2\theta$ (since we are working in P, whose HHS structure has been normalised). If $U \perp U_i$ for all i, then $\pi_U(H)$ uniformly coarsely coincides with $\pi_U(z_0)$, so it is bounded. Hence $d_U(\mathfrak{g}_H(x_1),\mathfrak{g}_H(x_2)) \leqslant \mathsf{diam}(\pi_U(H)) + 2B_0 \leqslant T_0 + 2B_0$ for all such U.

On the other hand, if $U \in \mathfrak{U}$ (i.e. $U \sqsubseteq V$ for some $V \in \operatorname{supp}(p^\pm)$) and $x \in \overline{P}$, then $d_U(\pi_U(x),\pi_U(H))$ is uniformly bounded, so $\pi_U(x)$ is uniformly close to $\pi_U(\mathfrak{g}_H(x))$. Thus, up to uniformly enlarging B_0, we have $d_U(x,\mathfrak{g}_H(x)) \leqslant B_0$ for such U and x. Hence

$$d_U(x_1,x_2) - 2B_0 \leqslant d_U(\mathfrak{g}_H(x_1),\mathfrak{g}_H(x_2)) \leqslant d_U(x_1,x_2) + 2B_0$$

for $U \in \mathfrak{U}$.

By definition, $d_H(\mathfrak{g}_H(x_1),\mathfrak{g}_H(x_2)) = d_P(\mathfrak{g}_H(x_1),\mathfrak{g}_H(x_2))$. By the strong distance formula (Theorem A.1), applied to $(H,\mathfrak{S}_H)$, there is a uniform $T \geqslant T_0+2B_0$ such that for all $T' \geqslant T$, we have L'_H (depending uniformly on T') such that

$$\begin{aligned}\sum_{U \in \mathfrak{S}} \{\!\{d_U(\mathfrak{g}_H(x_1),\mathfrak{g}_H(x_2)) + 2B_0\}\!\}_{T'} &= \sum_{U \in \mathfrak{U}} \{\!\{d_U(\mathfrak{g}_H(x_1),\mathfrak{g}_H(x_2)) + 2B_0\}\!\}_{T'} \\ &\leqslant L'_H d_H(\mathfrak{g}_H(x_1),\mathfrak{g}_H(x_2)) + L'_H,\end{aligned}$$

where we used the description of $\mathfrak{S}_H$ from Remark 2.29 and the above bound on the distance formula terms corresponding to $U \in \mathfrak{S} - \mathfrak{U}$. The claim follows since $d_U(x_1,x_2) \leqslant d_U(\mathfrak{g}_H(x_1),\mathfrak{g}_H(x_2)) + 2B_0$ for $U \in \mathfrak{U}$. ∎

Claim 4. There exists T'', depending only on the HHS parameters of $(P, \mathfrak{S})$, such that for all $T' \geqslant T''$, there exists $\check{L}$, depending only on the HHS parameters of $(P, \mathfrak{S})$ and T', such that

$$\sum_{U \in \mathfrak{S} - \mathfrak{U}} \{\!\{ d_U(x_1, x_2) \}\!\}_{T'} \leqslant \check{L} d_{\check{P}}(c(x_1), c(x_2)) + \check{L}$$

for all $x_1, x_2 \in P$.

Proof of Claim 4. The pair $(\check{P}, \mathfrak{S} - \mathfrak{U})$ is an HHS with uniform parameters, by Corollary 2.43, with the projections being specified in the proof of said corollary. Theorem A.1, applied to this HHS structure, yields the claim. ■

Both $\mathfrak{g}_H$ and c are uniformly coarsely lipschitz, coarsely surjective and quasi-median maps. To get the other bound needed to show that $c|_{\overline{P}} \times \mathfrak{g}_H|_{\overline{P}}$ is a quasi-isometry, let T, T'' be the uniform constants from Claims 3 and 4, let $T' \geqslant \max\{T, T''\}$, let $L_H, \check{L}$ be the constants provided by those claims, for the given T', and let $L := \max\{L_H, \check{L}\}$. We can moreover assume that T' is sufficiently large to be a threshold in the distance formula for $(P, \mathfrak{S})$ (with the affine function in Theorem A.1 taken to be the identity). Then Theorem A.1 yields a uniform constant L' such that, for all $x_1, x_2 \in \overline{P}$,

$$\begin{aligned}
d_X(x_1, x_2) &\leqslant L' \sum_{U \in \mathfrak{S}} \{\!\{ d_U(x_1, x_2) \}\!\}_{T'} + L' \\
&= L' \left[\sum_{U \in \mathfrak{S} - \mathfrak{U}} \{\!\{ d_U(x_1, x_2) \}\!\}_{T'} + \sum_{U \in \mathfrak{U}} \{\!\{ d_U(x_1, x_2) \}\!\}_{T'} \right] + L' \\
&\leqslant L' \left[L \left(d_{\check{P}}(c(x_1), c(x_2)) + d_H(\mathfrak{g}_H(x_1), \mathfrak{g}_H(x_2)) \right) + 2L \right] + L'.
\end{aligned}$$

Thus $c|_{\overline{P}} \times \mathfrak{g}_H|_{\overline{P}}$ is a quasi-median quasi-isometry with constants depending only on the HHS parameters. Now let $\xi := \Phi \circ c|_{X(A)}$ and $\eta := \Psi \circ \mathfrak{g}_H|_{X(A)}$. The maps Φ, Ψ and $c|_{\overline{P}} \times \mathfrak{g}_H|_{\overline{P}}$ are all A–equivariant uniform quality quasi-isometries, so $\xi \times \eta$ is an A–equivariant uniform quality quasi-isometry. This proves items (6) and (7).

Since $c(X(A))$ is uniformly EHQC in $\check{P}$ and H is uniformly HQC in P, it also follows from this that $X(A)$ is uniformly EHQC. To see that $X(A)$ is a coarse lipschitz retract, define a map $X \to X(A)$ by first applying the gate map to $\overline{P}$, then applying the quasi-isometry to $\check{P} \times H$, followed by the map $\rho \times \mathrm{id}_H$, where ρ is the coarsely lipschitz retraction $\check{P} \to Z(A, \check{P}, k)$ given by Proposition 4.1.(4). By perturbing the resulting map uniformly so that it is the identity on $X(A)$, we obtain a uniformly coarse lipschitz retraction $X \to X(A)$. We have thus proved item (8). □

If $H \leqslant G$ is a pair of groups, we denote the commensurator of H in G by $\mathrm{Comm}_G(H)$: i.e. $\mathrm{Comm}_G(H)$ is the subgroup of G consisting of all $g \in G$ such that $gHg^{-1} \cap H$ has finite index in both H and gHg^{-1}. In the next lemma, we continue to work with an HHS $(X, \mathfrak{S})$ and a finitely generated virtually abelian group $A \leqslant \mathrm{Isom}(X)$ acting hierarchically semisimply by HHS automorphisms.

Lemma 5.7. *Suppose $A \leqslant B$, and the action of A extends to an action of B on X by HHS automorphisms. Then the following hold:*

(1) Suppose $\mathrm{Comm}_B(A) = B$. Then $\mathrm{supp}(p^{\pm})$ is B–invariant. Thus $P \subseteq X$ is B-invariant and there is a finite-index subgroup $B' \leqslant B$ which fixes each $U \in \mathrm{supp}(p^{\pm})$ and fixes $\pi_U(p^{\pm}) \in \partial \mathcal{C}U$ for each $U \in \mathrm{supp}(p^{\pm})$.

(2) *Suppose* $\mathrm{Comm}_B(A) = B$. *Then* $\mathfrak{U}$ *is* B*–invariant. Hence the* B*–action on* $(X, \mathfrak{S})$ *induces an action of* B *on* $(\check{P}, \mathfrak{S} - \mathfrak{U})$ *by HHS automorphisms making* $c : P \to \check{P}$ *a* B*–equivariant map.*
(3) *If* A *is normal in* B*, then* B *stabilises* $X(A)$*.*

Proof. First assume that $\mathrm{Comm}_B(A) = B$. Let $V \in \mathfrak{S}$ and $x \in X$. Then $\pi_V(A \cdot x)$ is unbounded if and only if $V \in \mathrm{supp}(p^+)$, by Lemma 5.2.(4). The same holds when A is replaced by any finite-index subgroup. Hence, for any $b \in B$, we have that $\pi_V(bAb^{-1} \cdot x)$ is unbounded if and only if $V \in \mathrm{supp}(p^+)$, and also if and only if $V \in b\,\mathrm{supp}(p^+)$, so $\mathrm{supp}(p^+)$ is B–invariant. Thus there is a finite-index $B'' \leqslant B$ that fixes each of the finitely many $U \in \mathrm{supp}(p^+)$.

Let $b \in B''$, so b acts as an isometry of $\mathcal{C}U$ and hence a homeomorphism of $\partial\mathcal{C}U$. Let $A' \leqslant A$ be a finite index subgroup fixing all elements of $\mathrm{supp}(p^\pm)$ and fixing p^- and p^+. Then the A'–action on $\partial\mathcal{C}U$ has exactly two fixed points, namely $\pi_U(p^\pm)$, and the same is true of any finite-index subgroup. The fixed-point set of $bA'b^{-1}$ in $\partial\mathcal{C}U$ is $\{b\pi_U(p^+), b\pi_U(p^-)\}$. Since $bA'b^{-1} \cap A'$ has finite index in A', we therefore have that $b\pi_U(p^\pm) \in \{\pi_U(p^+), \pi_U(p^-)\}$. So, up to passing to a further finite-index subgroup, B'' fixes p^- and p^+, proving (1). Since $\mathrm{supp}(p^\pm)$ is B–invariant, and B preserves the $\sqsubseteq, \perp, \pitchfork$ relations in $\mathfrak{S}$, the set $\mathfrak{U}$ is B–invariant, which implies item (2), using Corollary 2.43. This argument also shows that the B–action on $\prod_{V\in\mathfrak{U}} \mathcal{C}V$ preserves the subset $\prod_{V\in\mathfrak{U}} \gamma_V$.

Now assume $A \trianglelefteq B$, so items (1) and (2) apply and B stabilises $\check{Z}$. Together with the B–invariance of $\prod_{V\in\mathfrak{U}} \gamma_V$, this shows that B stabilises $X(A)$. □

6. Hierarchical flat torus theorem

Now we can state and prove the main theorem:

Theorem 6.1. *Let* $(X, \mathfrak{S})$ *be an HHS of complexity* χ*. Then there exist constants* p, q *and* r*, depending only on the HHS parameters, such that the following holds. Let* $A \leqslant \mathrm{Isom}(X)$ *be a finitely-generated, infinite, virtually abelian group acting properly and hierarchically semisimply by HHS automorphisms on* $(X, \mathfrak{S})$*. Then:*

(1) *Let* n *be the maximal rank of free abelian subgroups of* A*. Then* $n \leqslant \chi$*.*
(2) *There exists a* (p, q)*–EHQC,* A*–invariant subspace* $F \subseteq X$*, a proper, cobounded action* $\alpha : A \to \mathrm{Isom}(\mathbb{E}^n)$*, and an* A*–equivariant* (p, p)*–quasi-isometry* $F \to \mathbb{E}^n$*, where* F *has the subspace metric from* X*.*
(3) *There is an injective uniformly coarse median metric space* (Y, d_Y) *and a CAT(0) space* (C, d_C) *with the following properties. Let* A *act on* $Y \times C \times \mathbb{E}^n$*, trivially on the first and second factors and via* α *on the third. Then there is a* (p, q)*–EHQC,* A*–invariant* r*–coarse lipschitz retract* $\mathcal{M}_A \subseteq X$ *and an* A*–equivariant* (p, p)*–quasi-isometry*
$$\Theta : \mathcal{M}_A \to Y \times C \times \mathbb{E}^n,$$
where $\mathcal{M}_A$ *has the metric inherited from* X*. Moreover, for each* $(y, c) \in Y \times C$*, the* A*–invariant subset*
$$F_{(y,c)} := \Theta^{-1}(\mathcal{N}_k((y, c)) \times \mathbb{E}^n)$$
is (p, q)*–EHQC and an* r*–coarse lipschitz retract.*
(4) *There exists* r_A*, depending on* A*, and a free abelian subgroup* $A_0 \trianglelefteq A$ *such that* $d_{Haus}(\mathcal{M}_{A_0}, X(A_0)) \leqslant r_A$*. Hence there exists* $\lambda \geqslant 0$*, depending on* A *and the HHS parameters, such that* $\mathcal{M}_{A_0}$ *is a* λ*–coarse median subalgebra of* X*.*
(5) *There is a function* $\omega : [1, \infty) \to [0, \infty)$*, depending on* A *and the HHS parameters, such that for all* $t \in [0, \infty)$*, if* $K \subseteq H_\theta(X(A))$ *is* A*–invariant and* (t, t)*–quasi-isometric to* $\mathbb{E}^n$*, then* $K \subseteq \mathcal{N}_{\omega(t)}(F_{(y,c)})$ *for some* $(y, c) \in Y \times C$*.*

Proof. By Lemma 2.47, Remark 2.48, and Lemma 2.49, we can and shall assume that X is a graph (and d is the usual graph metric) and A acts on X freely, so Proposition 5.6 applies.

Let $A_0 \trianglelefteq A$ be the intersection of a normal finite index free abelian subgroup of A with every subgroup of A with index $\leqslant \chi!$. Since A is finitely generated, this subgroup has finite index in A.

Let $p^+, p^- \in \partial X$ be the boundary points provided by Lemma 5.2. Recall the notation from Definitions 5.3 and 5.5. Let $k \geqslant k_1$, $M \geqslant M' \geqslant \theta$ and $L_1, L_2, L_3, L_4 \geqslant 1$ be the uniform constants from Proposition 5.6. Fix $z_0 \in X(A)$, let $H := A \cdot H_{M', z_0}(p^-, p^+)$, and let Y, Q be the spaces and $\Phi, \mathfrak{g}_H, \Psi, \xi, \eta$ be the maps produced by Proposition 5.6.

The action of A on X, and therefore on $H \subseteq X$ is proper so, since Ψ is a quasi-isometry, the A–action on (Q, d_Q) is proper. The median space Q is obtained by applying Theorem 3.1, so Proposition 3.2 applies: there exist median quasi-lines $Q_1, \ldots, Q_r$ such that A acts isometrically on $\prod_{i=1}^r Q_i$ and there is an A-equivariant isometry $Q \to \prod_{i=1}^r Q_i$. Since the rank of Q is at most χ by Lemma 3.19.(3), we have $r \leqslant \chi$. Thus A_0 acts trivially on the set of factors of $\prod_{i=1}^r Q_i$. Proposition 3.2 furthermore implies that the action of A_0 on each Q_i is cocompact. Since A_0 is free abelian, this also implies that A_0 fixes the endpoints of each Q_i. Identify Q with $\prod_{i=1}^r Q_i$.

CAT(0) minset in Q and proof of (1). Each Q_i is a finite rank, connected complete median space so [Bow16] provides a complete CAT(0) metric σ_i on Q_i, bilipschitz equivalent to d_{Q_i} (with uniform constant), such that any isometry of (Q_i, d_{Q_i}) is an isometry of (Q_i, σ_i). Let σ be the ℓ^2 product metric on Q, where each factor Q_i is equipped with σ_i. Then σ is a complete CAT(0) metric on Q, the action $A \curvearrowright Q$ is by σ-isometries, and the identity map $(Q, d_Q) \to (Q, \sigma)$ is bilipschitz with uniform constant (in fact, one can take $\sqrt{\chi}$ as the bilipschitz constant). In particular, the A–action on (Q, σ) is also proper.

Since the A–action on X is hierarchically semisimple, each $\langle a \rangle \leqslant A$ acting with unbounded orbits has the property that a has positive stable translation length on X (combine Definition 5.1 with the fact that π_U is coarsely lipschitz for each $U \in \mathfrak{S}$), from which it follows that the A–action on (Q, σ) is semisimple.

Hence [BH99, II.7.1] implies that the minset

$$\mathrm{Min}_Q(A_0) := \{x \in Q : \sigma(x, ax) = \inf_{y \in Q} \sigma(y, ay) \text{ for all } a \in A_0\}$$

is nonempty, convex, A–invariant and isometric to a product $D \times \mathbb{E}^n$, where n is the rank of A_0. The action of A on $D \times \mathbb{E}^n$ preserves the product decomposition, with A acting properly and cocompactly on $\mathbb{E}^n$, and elliptically on D, since the finite index subgroup A_0 acts trivially on D. We henceforth identify $\mathrm{Min}_Q(A_0)$ with $D \times \mathbb{E}^n$ in an A_0–equivariant way.

For any $x \in \mathbb{E}^n$, the subspace $D \times \{x\} \subseteq Q$ is closed and convex and – when equipped with the metric induced by σ – isometric to D. Therefore D is CAT(0). Let $C \subseteq D$ be the fixed point set of the A–action on D. In particular, C is a nonempty, closed, convex subset of D, so it is also CAT(0).

Since Q has rank at most χ by Lemma 3.19, any locally compact subset of Q has topological dimension at most χ by [Bow18b, Lem. 4.1]. So, since $\mathbb{E}^n$ is bilipschitz–embedded in (Q, d_Q), we get $n \leqslant \chi$, proving (1).

Description of $\mathcal{M}_A$ and Θ and proof of (4). Combining Proposition 5.6.(7) with the fact that (Q, d_Q) and (Q, σ) are $\sqrt{\chi}$–bilipschitz equivalent, the product map $\xi \times \eta : X(A) \to Y \times (Q, \sigma)$ is an (R, R)–quasi-isometry for some uniform R. We can assume R is large enough that η is R–coarsely surjective with respect to σ. Let

$$\mathcal{M}_A := \{x \in X(A) : \sigma(\eta(x), C \times \mathbb{E}) \leqslant R\}.$$

Since the closest point projection in (Q, σ) from $\mathcal{N}_R(C \times \mathbb{E})$ to the CAT(0) convex set $C \times \mathbb{E}$ is an A–equivariant uniform quasi-isometry (it moves each point a uniformly bounded distance), we can perturb $\xi \times \eta$ to an equivariant, p–coarsely surjective, (p, p)–quasi-isometry $\Theta : \mathcal{M}_A \to Y \times C \times \mathbb{E}^n$, where p is uniform.

Let $r_0 \geqslant 0$ be chosen (depending on A_0 and the HHS parameters) so that for all i, the space Q_i is contained in the r_0–neighbourhood of any A_0–orbit in Q_i. Hence $\sigma_i(\mathrm{Min}_{Q_i}(A_0), x) \leqslant r_0$ for all $x \in Q_i$. Therefore, since, by the definition of σ, we have $\mathrm{Min}_Q(A_0) = \prod_{i=1}^r \mathrm{Min}_{Q_i}(A_0)$, there exists $r_1 \geqslant 0$, depending only on A_0 and the HHS parameters, such that all of Q is contained in the r_1–neighbourhood of $\mathrm{Min}_Q(A_0)$. Let $x \in X(A)$. Choose $x' \in \mathrm{Min}_Q(A_0)$ such that $\sigma(x', \eta(x)) \leqslant r_1$ and let $y := \xi(x)$. Then choose $m \in \mathcal{M}_{A_0}$ such that $\xi \times \eta(m)$ is p–close in $Y \times \mathrm{Min}_Q(A_0)$ to (y, x'). Then

$$\begin{aligned} d_X(m, x) &\leqslant R(d_{Y\times Q}(\xi \times \eta(m), \xi \times \eta(x)) + R) \\ &\leqslant R(k + r_1 + R). \end{aligned}$$

This proves (4).

Proof of (2)-(3). Let us show $\mathcal{M}_A$ is uniformly EHQC. Let $x, x' \in \mathcal{M}_A$ and let $z, z' \in C \times \mathbb{E}$ be such that $\Theta(x) = (c(x), z)$ and $\Theta(x') = (c(x'), z')$. Let $\check{\alpha}$ be a uniform-quality hierarchy path in $\check{P}$ which joins $c(x), c(x')$ and lies in $c(X(A))$. As in the proof of Proposition 4.1, the σ–geodesic β in $C \times \mathbb{E}$ joining z to z' is a geodesic with respect to d_Q. Let α_1 be the path in $Y \times Q$ given by $\alpha_1(t) = (\check{\alpha}(t), z)$ for all t, and let β_1 be the path in $Y \times Q$ given by $\beta_1(t) = (c(x'), \beta(t))$ for all t. Let γ_1 be the concatenation of α_1 and β_1. Note that α_1 and β_1 lie in $Y \times C \times \mathbb{E}$. Hence, for each t, there exists $\gamma(t) \in \mathcal{M}_A$ such that $d_{Y\times Q}(\Theta(\gamma(t)), \gamma_1(t)) \leqslant p$. The path $t \mapsto \gamma(t)$ is a uniform quality quasigeodesic in $X(A)$ and, after a uniform perturbation, joins x to x'. The map η is uniformly quasimedian by Proposition 5.6, and c is uniformly quasimedian by Corollary 2.43. This implies that γ is a uniform quality hierarchy path, since α is a uniform quality hierarchy path in $\check{X}$ and β is a geodesic in the median metric d_Q. By construction, γ is a path in $\mathcal{M}_A$.

To see that $\mathcal{M}_A$ is a coarse lipschitz retract define a map $X \to \mathcal{M}_A$ by composing the uniformly coarse lipshcitz retract $X \to X(A)$ given by Proposition 5.6.(7) with the quasi-isometry $\xi \times \eta : X(A) \to Y \times (Q, \sigma)$, followed by the map $\mathrm{id} \times \pi_{C\times\mathbb{E}^n} : Y \times Q \to Y \times C \times \mathbb{E}^n$, where id is the identity on Y and $\pi_{C\times\mathbb{E}^n}$ is the closest point projection from Q onto $C \times \mathbb{E}^n$ (which is well-defined and 1-lipschitz because $C \times \mathbb{E}^n$ is convex and (Q, σ) is CAT(0)). Finally, compose with a quasi-inverse of $\xi \times \eta$. The resulting map is uniformly coarsely lipschitz since it is a composition of uniformly coarsely lipschitz maps, and it can be uniformly perturbed so that its image is contained in $\mathcal{M}_A$ and it is the identity on $\mathcal{M}_A$.

Now fix $c \in C$ and $y \in Y$. Let $F_{(y,c)}$ be the set of $m \in \mathcal{M}_A$ such that $\Theta(m)$ is p–close to the subset $\{(y, c)\} \times \mathbb{E}^n$ of $Y \times Q$. Since Θ is A–equivariant and A acts trivially on $Y \times C$, the set $F_{(y,c)}$ is A–invariant, and the choice of p ensures that Θ restricts to a uniformly coarsely surjective, uniform quasi-isometry from $F_{(y,c)}$ to $B \times \mathbb{E}^n$, where B is a uniformly bounded set where A acts trivially. So composing with the natural projection $B \times \mathbb{E}^n \to \mathbb{E}^n$ gives a uniform quasi-isometry $\tau : F_{(y,c)} \to \mathbb{E}^n$.

Let $\alpha : A \to \mathrm{Isom}(\mathbb{E}^n)$ be induced by projecting the action on $Y \times C \times \mathbb{E}^n$ to the third factor. Then τ is A–equivariant.

To complete the proof of (2) and (3), we therefore just have to observe that $F_{(y,c)}$ is uniformly EHQC and a uniformly coarse lipschitz retract, by essentially the same argument as was used for $\mathcal{M}_A$. Indeed, if $x, x' \in F_{(y,c)}$, then $c(x), c(x')$ are uniformly close, so the hierarchy path γ produced above stays, by construction, in a uniformly bounded neighbourhood of $F_{(y,c)}$. Moreover, the construction of a coarse lipschitz map $X \to \mathcal{M}_A$ above can be reproduced using the closest point projection $Q \to \{c\} \times \mathbb{E}^n$ rather than $Q \to C \times \mathbb{E}^n$ (since the former is also convex in (Q, σ)) resulting, up to a uniform perturbation, in a uniform quality coarsely lipschitz retraction $X \to F_{(y,c)}$.

Proof of item (5) Let $\overline{H}_\theta(X(A))$ be the θ-hull of $X(A)$ in $\overline{P}$. It follows from Proposition 5.6 that there exists $L_3' \geqslant 1$, depending only on L_3, M and the HHS parameters, such that $\overline{H}_\theta(X(A))$ is (L_3', L_3')–quasi-isometric to $H_\theta(\check{Z}) \times H$. The Hausdorff distance between $\overline{H}_\theta(X(A))$ and $H_\theta(X(A))$ is uniformly bounded, so there exists L depending only on the HHS parameters such that $H_\theta(X(A))$ is (L, L)-quasi-isometric to $H_\theta(\check{Z}) \times H$.

Given $t \geqslant 1$, let $\mathcal{K}_t$ be the set of A–invariant subspaces $K \subseteq H_\theta(X(A))$ such that there is a (t,t)–quasi-isometry $K \to \mathbb{E}^n$. Let

$$\omega(t) = \sup\{\sup_{x_0 \in K} d(x_0, \mathcal{M}_A) : K \in \mathcal{K}_t\}.$$

Note that ω is a non-decreasing function of t since $\mathcal{K}_t \subseteq \mathcal{K}_{t'}$ for $t \leqslant t'$, so we need to show that ω takes finite values and is unbounded as $t \to \infty$.

Let $s_0 \in [1, \infty)$ and suppose $K \subseteq H_\theta(X(A))$ is A–invariant and there exists $x_0 \in K$ such that $d(x_0, \mathcal{M}_A) > s_0$. Then either

(1) $d_H(\mathfrak{g}_H(x_0), \mathfrak{g}_H(\mathcal{M}_A)) > s_1$ or
(2) $d_{\check{X}}(c(x_0), \check{Z}) > s_1$,

where $s_1 := \frac{s_0 - L}{2L}$.

By (4), there is a point $x' \in \mathcal{M}_{A_0}$ such that $d_X(\mathfrak{g}_H(x_0), x') \leqslant r_A$ and, by definition, there exists $z_0 = (d_0, e_0) \in \mathrm{Min}_Q(A_0) = D \times \mathbb{E}^n$ such that $\sigma(z_0, \Psi(x')) \leqslant R$.

Suppose that case (1) holds. Then there exists $\kappa \geqslant 1$, depending on L_2, r_A, R, χ such that $\sigma(d_0, C) \geqslant s_1/\kappa - \kappa$. Hence Lemma 6.2 below implies that there exists $d_1 \in D$ such that $d_1 \in A \cdot d_0$ and $d_D(d_0, d_1) \geqslant s_1/\kappa - \kappa$. In this case, let $a_1 \in A$ be such that $a_1 d_0 = d_1$ and let $x_1 := a_1 x_0$.

Now assume that case (2) holds. Let ω_0 be the increasing function from Proposition 4.1, applied to the action of A on $\check{X}$. Then $d_{\check{X}}(c(x_0), \check{Z}) > s_1$, so $\mathsf{diam}(A \cdot c(x_0)) > \omega_0(s_1)$, by Proposition 4.1.(5), and thus there is a point $y \in c(K)$ such that $d_{\check{X}}(c(x_0), y) \geqslant \omega_0(s_1)$. In this case, let $x_1 \in K$ be such that $c(x_1) = y$.

In either case, let $\Delta_i := A_0 \cdot x_i \subseteq K$ for $i \in \{0, 1\}$, which is a subspace of $H_\theta(X(A))$. Hence Δ_i is (L, L)-quasi-isometric to $c(\Delta_i) \times \mathfrak{g}_H(\Delta_i)$. Indeed, $c(\Delta_i)$ has uniformly bounded diameter by the choice of x_i, so the claimed quasi-isometry holds up to uniformly increasing L. By Proposition 4.1.(6), $\mathsf{diam}(c(\Delta_i)) = \mathsf{diam}(A_0 \cdot c(x_i))$ is uniformly bounded, and by construction $\mathfrak{g}_H(\Delta_i)$ is (L', L')–quasi-isometric to $\mathbb{E}^n$, with constant L' depending only on the HHS parameters and A. Hence there exists $L'' \geqslant 1$, depending only on the HHS parameters and A, and (L'', L'')-quasi-isometries $\Xi_i : \mathbb{E}^n \to \Delta_i$.

Suppose that $K \in \mathcal{K}_t$ for some $t \geqslant 1$, and suppose that $\zeta : K \to \mathbb{E}^n$ is a (t,t)–quasi-isometry. Then $\zeta \circ \Xi_i : \mathbb{E}^n \to \mathbb{E}^n$ is a $(t(L''+1), t(L''+1))$–quasi-isometric embedding for $i \in \{0,1\}$. Hence, by Lemma 6.3, each of the two maps $\zeta \circ \Xi_i$ is $C'\,(t(L''+1))$–coarsely surjective. Hence there exist $x_i' \in \Delta_i$ such that $|\zeta(x_0') - \zeta(x_1')| \leqslant C'\,(t(L''+1))$, and therefore $d_X(\Delta_0, \Delta_1) \leqslant L''(C'\,(t(L''+1)) + L'')$.

On the other hand, in either of the above cases, we have shown that $d_X(\Delta_0, \Delta_1) \geqslant \min\{\kappa' s_0 - \kappa', \kappa' \omega_0(s_0 - L)/2L - \kappa'\}$, where κ' depends only on the HHS parameters and A (but not on K). Hence

$$\min\{\kappa' s_0 - \kappa', \kappa' \omega_0(s_0 - L)/2L - \kappa'\} \leqslant L''(C'\,\big(t(L''+1)\big) + L''),$$

so since ω_0 is an ubounded, increasing function, it follows that $s_0 \leqslant \omega(t)$ where $\omega : [1, \infty) \to [1, \infty)$ is a function that can be chosen in terms of the HHS parameters and A only.

Thus far, we have shown $K \subseteq \mathcal{N}_{\omega(t)}(\mathcal{M}_A)$ whenever $K \in \mathcal{K}_t$. The above argument also showed that $c(K)$ has diameter in $\check{X}$ bounded above by a function of t that can be chosen in terms of the HHS parameters and A. Similarly, the image of K in C is bounded uniformly in terms of t. Therefore, $\Theta(K)$ has diameter bounded uniformly in terms of t, so, up to uniform perturbation of ω, we get that $K \subseteq \mathcal{N}_{\omega(t)}(F_{(y,c)})$ for some $y \in Y, c \in C$. $\square$

We used the following standard lemmas in the above proof.

Lemma 6.2. *Let D be a CAT(0) space, let G be a group acting elliptically on D and let $\mathrm{Fix}(G) \subseteq D$ be the fixed point set of G. Then $\mathsf{diam}(G \cdot x) \geqslant d(x, \mathrm{Fix}(G))$ for all $x \in D$.*

Proof. Fix $x \in D$. Let r be the infimal radius of balls containing the bounded set $G \cdot x$; note that $r \leqslant \mathsf{diam}(G \cdot x)$. Then [BH99, Prop. II.2.7] provides a unique circumcentre for $G \cdot x$, i.e. a point $c \in D$ such that $G \cdot x \subset \mathcal{N}_r(c)$. Since $g \cdot c$ has this property for all $g \in G$, uniqueness of c implies that $c \in \mathrm{Fix}(G)$, so $d(x, \mathrm{Fix}(G)) \leqslant d(x, c) \leqslant r \leqslant \mathsf{diam}(G \cdot x)$. $\square$

Lemma 6.3. *For each $n \in \mathbb{N}$, there is a function $C' : [1, \infty) \to [0, \infty)$ such that the following holds. Let $f : \mathbb{E}^n \to \mathbb{E}^n$ be a (C, C)–coarsely lipschitz map. Suppose that $\sup_{x \in \mathbb{E}^n} \mathsf{diam}(f^{-1}(\mathcal{N}_r(x))) < \infty$ for all $r \geqslant 0$. Then f is $C'(C)$–coarsely surjective.*

Proof. By a standard triangulation argument, there is a continuous map $g : \mathbb{E}^n \to \mathbb{E}^n$ such that $\sup_{x \in \mathbb{E}^n} |f(x) - g(x)| \leqslant R$, and g is (K, K)–coarsely lipschitz, for some R, K depending only on C. We will show that g is surjective, from which it will follow that f is R–coarsely surjective.

Let $\bar{E}$ be the one-point compactification of $\mathbb{E}^n$. Since g is continuous and preimages of bounded sets are bounded, g is proper map. Hence g extends to a continuous map $\bar{g} : \bar{E} \to \bar{E}$ with $\bar{g}(\infty) = \infty$.

Define a homeomorphism $h : \bar{E} \to \mathbb{S}^n$ to be stereographic projection, where $\mathbb{E}^n$ is viewed as a coordinate plane in $\mathbb{E}^{n+1}$ and $\mathbb{S}^n$ is viewed as the sphere in $\mathbb{E}^{n+1}$ centred at the origin and with radius R, where $R = \sup_{x \in \mathbb{E}^n} \mathsf{diam}(g^{-1}(\mathcal{N}_1(x))) < \infty$. The "north pole" for the projection is in the direction normal to $\mathbb{E}^n$, and $\mathbb{S}^n$ is identified with $\bar{E}$ in such a way that the north pole is $\infty \in \bar{E}$.

This choice of h has the following property: if $x, y \in \mathbb{E}^n$ and $g(x) = g(y)$, then $h(x)$ and $h(y)$ are not antipodal.

If g were not surjective, then $h \circ \bar{g} \circ h^{-1} : \mathbb{S}^n \to \mathbb{S}^n$ has range properly contained in $\mathbb{S}^n$, to the Borsuk-Ulam theorem provides antipodal points $h(x), h(y) \in \mathbb{S}^n$ such that $\bar{g}(x) = \bar{g}(y)$. Since $\bar{g}^{-1}(\infty)$ has a single point, $x, y \in \mathbb{E}^n$ and $g(x) = g(y)$, a contradiction. $\square$

A consequence of Theorem 6.1.(3) we will need is:

Corollary 6.4. *Let $(X, \mathfrak{S})$ be an HHS on which the finitely generated virtually abelian group $A \leqslant \mathrm{Isom}(X)$ acts properly and hierarchically semisimply by HHS automorphisms. Then there exist k, r, s, depending only on the HHS parameters, such that the following holds. There is a proper left-invariant quasigeodesic metric d_A on A such that for all $x \in \mathcal{M}_A$, there is an A–equivariant (s, s)–quasi-isometric embedding $\phi_x : (A, d_A) \to \mathcal{M}_A$ such that $\phi_x(1) = x$. Moreover, $\phi_x(A)$ is contained in the (k, r)–EHQC, A–invariant (k, k)–quasiflat F from Theorem 6.1.(3).*

Proof. By Theorem 6.1.(2),(3), there is a fixed proper cocompact action $\alpha : A \to \mathrm{Isom}(\mathbb{E}^n)$ and a uniform constant k such that for all $x \in \mathcal{M}_A$, there is an A–equivariant (k, k)–quasi-isometry $q : F_x \to \mathbb{E}^n$, where $F_x \subseteq \mathcal{M}_A$ is an A–invariant, (k, r)–EHQC subspace containing x. Let $\Omega = Cay(A, A)$, which is a graph of diameter 1 equipped with a free A–action. Let

$\mathbf{E} = \mathbb{E}^n \times \Omega$, so that the diagonal A–action on $\mathbf{E}$ is free, metrically proper, and cobounded. Let d_A be the metric on A obtained by pulling back the subspace metric on the orbit $A \cdot (\vec{0}, 1_A) \subset \mathbf{E}$. Note that d_A is uniformly quasi-isometric to the pseudometric coming from the orbit map $A \to \mathbb{E}^n$ given by $a \mapsto \alpha(a) \cdot \vec{0}$.

We can assume that $q(x) = \vec{0}$. Define $\phi_x : (A, d_A) \to (F_x, d_X)$ by $\phi_x(a) = ax$. Then $q \circ \phi_x(a) = \alpha(a) \cdot \vec{0}$, which is to say that $q \circ \phi_x$ is the orbit map associated to α, which is, by construction, a quasi-isometry with uniform constants. Since q is a quasi-isometry with constants independent of x and A, it follows that ϕ_x is also. □

Theorem 6.5. *Let $(X, \mathfrak{S})$ be an HHS and let $B \leqslant \mathrm{Isom}(X)$ act by HHS automorphisms. Suppose $A \trianglelefteq B$ is a normal finitely generated infinite abelian subgroup, and that the action of A on $(X, \mathfrak{S})$ is proper and hierarchically semisimple. Then there are finite index subgroups $A' \leqslant A$ and $B' \leqslant B$ such that A' is central in B', and the element of $H^2(B'/A', A')$ corresponding to the central extension $A' \hookrightarrow B' \twoheadrightarrow B'/A'$ is represented by a bounded cocycle.*

Proof. Let $p^+, p^- \in \partial X$ be the boundary points provided by Lemma 5.2 and let $S := \prod_{U \in \mathrm{supp}(p^+)} \mathrm{hull}_U(p^-, p^+)$. Recall Definitions 5.3 and 5.5. Item (1) of Lemma 5.7 implies that $\mathrm{supp}(p^\pm)$ and P are B–invariant and there exists $B' \leqslant B$ which fixes each $U \in \mathrm{supp}(p^\pm)$ and $\pi_U(p^\pm) \in \partial CU$, and item (3) implies that $X(A)$ is B–invariant. In particular, there is a natural isometric action of B on $\prod_{U \in \mathrm{supp}(p^+)} \mathcal{C}U$, and this restricts to an action of B' on S which preserves the factors of the above product decomposition.

Normal implies virtually central. Let Y be the injective space, from Proposition 5.6, such that there is an A–equivariant quasi-isometry $\Phi : c(X(A)) \to Y$, where A acts on Y trivially. Recall from the proof of that proposition and the proof of Proposition 4.1 that Y is the fixed point set of the A–action on the injective hull $\mathrm{Env}(\check{P})$, and Φ is the restriction of the $\mathrm{Isom}(\check{P})$–equivariant isometric embedding $\check{P} \hookrightarrow \mathrm{Env}(\check{P})$. Normality of A in B implies that Y is B–invariant, so there is an action of B on Y such that Φ is B–invariant.

Let $A' \leqslant A \cap B'$ be a finite index, free abelian, characteristic subgroup of A. Note that A' is normal in B'.

Claim 1. The action of A' on S is proper, and if the B'–action on X is proper, then the diagonal action of B' on $Y \times S$ is proper.

Proof of Claim 1. Proposition 3.2 provides a constant ζ such that $\pi_V(X(A))$ has diameter at most ζ for all $V \in \mathfrak{U} - \mathrm{supp}(p^+)$.

Consider the action of B' on $X(A)$. Since $c|_{\overline{P}} \times \mathfrak{g}_H|_{\overline{P}}$ is a quasi-isometry by Proposition 5.6.(6), it follows from Theorem A.1 and the existence of the constant ζ that $c|_{X(A)} \times \prod_{U \in \mathrm{supp}(p^+)} \pi_U|_{X(A)} : X(A) \to Y \times S$ is a B'–equivariant quasi-isometry. Hence the A'–action on $Y \times S$ is proper and, if the B'–action on X is proper, then so is the B'–action on $Y \times S$. Since $A' \leqslant B'$ acts on Y trivially, it follows that the A'–action on S is proper. ■

Let $N = |\mathrm{supp}(p^+)|$, so $N \leqslant \chi$ and S is naturally quasi-isometric to $\mathbb{E}^N$. For each $U \in \mathrm{supp}(p^+)$, recall that $L_U := \mathrm{hull}_U(p^-, p^+)$ is a quasi-line equipped with a B'–action fixing the endpoints, so by Lemma 3.5 and its proof, for each such U there exists a quasi-action q_U of B' on $\mathbb{R}$ such that (using Claim 1):

- For all $b \in B'$, $q_U(b)$ has bounded orbits if and only if b has bounded orbits in L_U.
- The restriction of q_U to A' is an isometric action.
- The diagonal quasi-action of B' on $\mathbb{R}^N$ given by $\prod_U q_U$ restricts on A' to a proper action.
- The resulting diagonal quasi-action of B' on $Y \times \mathbb{R}^N$ is quasi-conjugate to the original action of B' on $Y \times S$.

Let $a \in A'$ and $b \in B'$. Then $[b, a] \in A'$, since A' is normal in B', and therefore $[b, a]$ acts trivially on Y. On the other hand, the displacement of $q_U([b, a])$ is bounded above independently of a, b, so by properness, there is a finite subset $\mathcal{F} \subseteq A'$ such that $[b, a] \in \mathcal{F}$ for all $b \in B'$, $a \in A'$. Since A' is residually finite, there is a finite-index subgroup of A' disjoint from $\mathcal{F} - \{1\}$. Hence, replacing A' by this finite index subgroup, A' is central in B'.

Bounded cocycle. So far, we have a central extension $1 \to A' \hookrightarrow B' \xrightarrow{\phi} \bar{B}' := B'/A' \to 1$, and a quasi-action of B' on the Euclidean space $\mathbb{E}^N$ whose restriction to A' is a proper affine isometric action.

Let $\mathbb{F} \subseteq \mathbb{E}^N$ be an A'–invariant affine subspace on which A' acts properly and cocompactly by translations, using that A' is free abelian. Without loss of generality, $0 \in \mathbb{F}$. Let $R \geqslant 0$ be chosen so that $\mathbb{F} \subseteq A' \cdot \mathcal{N}_R^{\mathbb{E}^N}(0)$.

Let $\delta' \geqslant 0$ be such that $\|bb' \cdot 0 - b \cdot 0 - b' \cdot 0\|_2 \leqslant \delta'$ for all $b, b' \in B'$. Define a map $q : B' \to A'$ as follows. Let $\pi : \mathbb{E}^N \to \mathbb{F}$ be the orthogonal projection. Fix $b \in B'$. If $b \in A'$, let $q(b) = b$. Otherwise, let $q(b)$ be some element of A' such that $\|q(b) \cdot 0 - \pi(b \cdot 0)\|_2 \leqslant R$.

Given $b, b' \in B'$, we have $\|q(bb')\cdot 0 - q(b)\cdot 0 - q(b')\cdot 0\|_2 \leqslant \|\pi(bb')\cdot 0 - \pi(b\cdot 0) - \pi(b'\cdot 0)\|_2 + 3R \leqslant \delta' + 3R$. Hence there exists $\delta \geqslant 0$ such that $|q(bb') - q(b) - q(b')|_{A'} \leqslant \delta$ for all $b, b' \in B'$, where $|-|_{A'}$ is a a word norm on A' coming from some finite generating set. In other words, q is an A'–valued quasimorphism which is the identity on A'.

Let $s_0 : \bar{B}' \to B'$ be a set-theoretic section of the quotient map ϕ, with $s_0(1) = 1$. Define a new section $s : \bar{B}' \to B'$ as follows. Given $\bar{b} \in \bar{B}'$, let $a(\bar{b}) = q(s_0(\bar{b}))$, and let $s(\bar{b}) = s_0(\bar{b})a(\bar{b})^{-1} = a(\bar{b})^{-1}s_0(\bar{b})$. Then $\phi(s(\bar{b})) = \phi(s_0(\bar{b})) = \bar{b}$, so s is again a section of ϕ. But now $\|q(s(\bar{b}))\|_1 \leqslant \delta$ for all $\bar{b}$. Now, since s is a section of ϕ, the cohomology class corresponding to the central extension B' of B'/A' is represented by the cocycle $\bar{B'}^2 \ni (\bar{g}, \bar{h}) \mapsto s(\bar{g})s(\bar{h})s(\bar{g}\bar{h})^{-1} \in A'$. Hence $|q(s(\bar{g})s(\bar{h})s(\bar{g}\bar{h})^{-1})|_{A'} \leqslant |q(s(\bar{g})) + q(s(\bar{h})) - q(s(\bar{g}\bar{h}))|_{A'} + 3\delta \leqslant 6\delta$ for all $\bar{g}, \bar{h} \in \bar{B}'$, so the cocycle is bounded. □

7. Applications

We now describe some applications to groups acting on HHSes.

Definition 7.1. Let Γ be a group and let $A \leqslant \Gamma$ be a virtually abelian subgroup. Then A is *highest* if for all finite-index subgroups $A' \leqslant A$ and all virtually abelian subgroups $A'' \leqslant G$ with $A' \leqslant A''$, we have $[A'' : A'] < \infty$. ✽

Lemma 7.2. *Let $(X, \mathfrak{S})$ be an HHS and suppose G acts uniformly properly and hierarchically semisimply by isometric HHS automorphisms on $(X, \mathfrak{S})$. Then there are only finitely many abstract isomorphism types of finitely generated virtually abelian subgroups of G. In particular, there are constants B, N_0, R_0, depending on G but not on the specific choice of $(X, \mathfrak{S})$, such that for all finitely generated virtually abelian $A \leqslant G$,*

- $|A| \leqslant B$ *if* $|A| < \infty$*;*
- *there is a free abelian subgroup $A_0 \leqslant A$ with $[A : A_0] \leqslant N_0$;*
- $\mathrm{rk}(A) \leqslant R_0$.

Proof. Uniform properness provides a function $f : [0, \infty) \to [0, \infty)$ such that $|\{g \in G : d_G(x, gx) \leqslant r\}| \leqslant f(r)$ for all $x \in X$ and all $r \geqslant 0$. Corollary 2.46 provides a constant B', depending only on the HHS parameters, so that any finite subgroup $F \leqslant G$ has some orbit in X of diameter at most B'. Hence $|F| \leqslant f(B')$ for all finite subgroups $F \leqslant G$, which proves the first assertion. Let $\{F_1, \ldots, F_p\}$ be a set of finite subgroups containing exactly one representative of each abstract isomorphism class of finite subgroups of G.

If χ_0 is the minimal complexity of HHSs admitting G–actions satisfying the hypotheses of the lemma, then Theorem 6.1.(1) implies that $\mathrm{rk}(A) \leqslant \chi_0$ for all finitely generated virtually abelian subgroups of G. In fact, Proposition 5.6 together with properness (see also [HRSS25, Prop. 2.17] shows that $\mathrm{rk}(A)$ is bounded above by the maximal r such that X contains an r–dimensional quasiflat. So, Lemma 2.51 implies that A fits into an exact sequence

$$1 \to F_s \to A \to C \to 1$$

for some s, where C is a crystallographic group of dimension at most χ_0. Lemma 2.52 then implies that there are only finitely many possibilities for the isomorphism type of A. □

Corollary 7.3 (Ascending chain condition)**.** *Let G be a group and let $H_1 \leqslant H_2 \leqslant \cdots \leqslant G$ be an ascending chain of finitely generated virtually abelian subgroups. If G acts uniformly properly and hierarchically semisimply by isometric HHS automorphisms on an HHS $(X, \mathfrak{S})$, then $H_n = H_{n+1}$ for all sufficiently large n.*

In particular, any virtually abelian subgroup $H \leqslant G$ is finitely generated and virtually contained in a highest virtually abelian subgroup.

Proof. Let $H_1 \leqslant H_2 \leqslant \ldots$ be an ascending chain of finitely generated virtually abelian subgroups. Let R_0, N_0 be the constants from Lemma 7.2. After passing to a subsequence, there exists $m \in \{0, \ldots, R_0\}$ such that for each $n \geqslant 1$, there is a normal subgroup $A_n \leqslant H_n$ such that $A_n \cong \mathbb{Z}^m$ and $[H_n : A_n] \leqslant N_0$. Lemma 7.2 shows that we are done if $m = 0$, so assume $m \geqslant 1$.

For all n, $[H_1 : H_1 \cap A_n] \leqslant [H_n : A_n] \leqslant N_0$ and $H_1 \cap A_n$ is free abelian. Since H_1 has finitely many subgroups with index $\leqslant N_0$, we can assume, up to passing to a subsequence, that $A_1 = H_1 \cap A_n$ for all $n \in \mathbb{N}$. It therefore suffices to show that $[A_n : A_1]$, is bounded independently of n, since this will imply $H_n = H_{n+1}$ for sufficiently large n.

Let $X(A_1)$ be as in Proposition 5.6. Let $\mathcal{M}_{A_1} \subseteq X(A_1)$ be the subspace obtained by applying Theorem 6.1.(3) to A_1. The same theorem provides $q \geqslant 0$, depending only on A_1 and the HHS parameters, such that for all $y \in \mathcal{M}_{A_1}$, there is an A_1–invariant (k,k)–quasiflat $F(y) \subseteq \mathcal{M}_{A_1}$ and $F(y) \subseteq A_1 \cdot \mathcal{N}_q(y)$.

For any $n \geqslant 1$, we have $A_1 \leqslant A_n$ and since A_n is abelian, A_1 is normal in A_n. Hence A_n stabilises $X(A_1)$, by Lemma 5.7, and therefore each A_n stabilises the uniformly HQC subset $H_\theta(X(A_1))$. So, by Lemma 2.28, $H_\theta(X(A_1))$ is an HHS on which each A_n acts properly and hierarchically semisimply by HHS automorphisms. Now apply Theorem 6.1.(2) to this action, so that for each n, we get an A_n–invariant subspace F_n of $H_\theta(X(A_1))$ that is (k,k)-quasi-isometric to $\mathbb{E}^m$. Now, Theorem 6.1.(5) implies that $F_n \subseteq \mathcal{N}_{\omega(k)}(\mathcal{M}_{A_1})$, where ω is the function provided by Theorem 6.1.(5).

Choose $y \in \mathcal{M}(A_1)$ such that $d(y, y') \leqslant \omega(k)$ for some $y' \in F_n$. Define $\pi_n : F(y) \to F_n$ as follows. Given $x \in F(y)$, choose $a \in A_1$ such that $x \in \mathcal{N}_q(a \cdot y)$, and then let $\pi_n(x) = a \cdot y'$. Note that $d(x, \pi_n(x)) \leqslant q + \omega(k)$. So, π_n is C–coarsely lipschitz, where C depends on k, q, ω but not on n. Lemma 6.3 provides C', depending only on C and k, such that π_n is C'–coarsely surjective.

Now let $z \in F_n$. Choose $x \in F(y)$ such that $d(z, \pi_n(x)) \leqslant C'$. Then $d(z,x) \leqslant C' + q + \omega(k)$. Hence $F_n \subseteq A_1 \cdot \mathcal{N}_{C'+q+\omega(k)}(y)$. Thus $[A_n : A_1] \leqslant f(2(\omega(k) + q + C'))$, so $[H_n : A_1] \leqslant N_0 f(2(\omega(k) + q + C'))$ which is bounded independently of n, as needed. □

Corollary 7.4. *Let $(X, \mathfrak{S})$ be an HHS and let G be a group acting uniformly properly and hierarchically semisimply by HHS automorphisms on X. If G is virtually solvable then G is virtually abelian.*

Proof. Suppose G is virtually solvable with derived length $k \geqslant 1$. We proceed by induction on k. If $k = 1$ then G is abelian, so suppose $k > 1$ and $G' := [G, G]$ is virtually abelian. It follows from Corollary 7.3 that G' is finitely generated, so there exists a finite index, finitely generated, free abelian subgroup $A \leqslant G'$. By Theorem 6.5, we can assume, up to taking a further finite index subgroup, that A is central in a finite index subgroup $H \leqslant G$. Moreover the element of $H^2(H/A, A)$ corresponding to the central extension $1 \to A \to H \to H/A \to 1$ is bounded. This implies that H is quasi-isometric to $A \times H/A$, by [Ger92]. Let $\pi : H \to H/A$ be the quotient map and let $H' := [H, H]$. Since $H' \leqslant G'$, the image $\pi(H')$ is finite. Moreover $\pi(H')$ is normal in H/A and $(H/A)/\pi(H') = H/H'$ is abelian, so [BH99, II.7.9] implies that H/A is virtually abelian. Therefore $A \times H/A$ is virtually abelian and, since virtually abelian groups are quasi-isometrically rigid [Pan83] (see [Dru09, Sec. 4.3]), this implies that H is virtually abelian. □

As we remarked earlier, the hierarchical semisimplicity assumption holds for HHGs, so the theorems of Section 6 apply to HHGs, yielding the following "coarse flat torus" result:

Corollary 7.5. *Let $(G, \mathfrak{S})$ be an HHG with complexity χ. There exists a constant k, depending only on the HHS parameters, such that for any virtually abelian subgroup $A \leqslant G$, there is an A–invariant (k,k)–EHQC subspace $F \subseteq G$, a proper, cocompact, isometric A–action on $\mathbb{E}^n$ for some $n \in \{0, \dots \chi\}$, and an A–equivariant (k,k)–quasi-isometry $F \to \mathbb{E}^n$.*

In particular, virtually abelian subgroups of G are finitely generated and belong to one of finitely many abstract isomorphism types.

Proof. Hierarchical semisimplicity comes from [DHS20, Thm. 3.1], at which point Theorem 6.1 applies, since A is finitely generated by Corollary 7.3. This gives the subspaces F from the statement, while the statement about isomorphism types comes from Lemma 7.2. □

7.1. **Centralisers, normalisers and commensurators.** We can now use the previous results to study the algebraic and geometric properties of the centralisers, normalisers and commensurators of virtually abelian subgroups of HHGS.

Corollary 7.6. *Let $(G, \mathfrak{S})$ be an HHG. Let $A \leqslant G$ be a virtually abelian subgroup. Then $N_G(A)$ lies at finite Hausdorff distance from the set $\mathcal{M}_A$ from Theorem 6.1.(3), and in particular is EQHC. Moreover, $[N_G(A_0) : C_G(A_0)] < \infty$, where A_0 is the finite-index subgroup of A from Theorem 6.1.(4).*

Finally, there is a free abelian subgroup $A_1 \leqslant A$ such that $C_G(A_1)$ is HQC in any HHG structure on G, and $[A : A_1] \leqslant D$, where $D \in \mathbb{N}$ depends only on the group G (and not on A or the choice of HHG structure).

Proofs of Corollary 7.6 and Corollary 1.7. By Corollary 7.5, the virtually abelian subgroup A is finitely generated. Let $n := \mathrm{rk}(A)$. The set $X(A)$ from Theorem 6.1 is $N_G(A)$–invariant by Lemma 5.7.(3). Recall from Theorem 6.1 the EHQC set $\mathcal{M}_A$, which is A–invariant. Let $\{x_i\}_{i \in I}$ contain exactly one point in each A–orbit in $\mathcal{M}_A$. Let $\mathcal{S}$ be a fixed finite generating set of A.

Let (A, d_A) be the metric group from Corollary 6.4. The same corollary then provides $k, r \geqslant 1$, depending only on the HHS parameters, such that for all $x \in \mathcal{M}_A$, there is an A–equivariant (r,r)–quasi-isometric embedding $\phi_x : (A, d_A) \to \mathcal{M}_A$ given by $\phi_x(a) = ax$, and $\phi_x(A) \subseteq F_x$, an A–invariant (k,k)–EHQC (k,k)–quasiflat in $\mathcal{M}_A$.

Claim 1. There exists $R < \infty$ such that $\mathcal{M}_A \subseteq \mathcal{N}_R(N_G(A))$. If A is abelian, the same holds with $N_G(A)$ replaced by $C_G(A)$.

Proof of Claim 1. Let $\xi = r \max_{s \in \mathcal{S}} d_A(1, s) + r$. Then $\phi_{x_i}(s) = sx_i \in \mathcal{N}_\xi(x_i)$ for all $i \in I$ and $s \in \mathcal{S}$. For each $i \in I$, we have $x_i^{-1}\mathcal{N}_\xi(x_i) = \mathcal{N}_\xi(1)$, which is finite. The inner automorphism of G given by conjugation by x_i^{-1} restricts to an injective map $\mathcal{S} \to \mathcal{N}_\xi(1)$. Since there are finitely many possibilities for this map, there is a finite set $\{i_1, \dots, i_p\} \subseteq I$ such that for all $i \in I$, there exists $j \leqslant p$ for which $x_{i_j} x_i^{-1} s x_i x_{i_j}^{-1} = s$ for all $s \in \mathcal{S}$. Hence $x_i = b x_{i_j}$ for some $b \in C_G(A)$. Now, any $x \in \mathcal{M}_A$ has the form ax_i for some $a \in A$ and $i \in I$, so

$x = abx_{i_j} \in N_G(A) \cdot x_{i_j}$. Letting $R = \max_{j \leqslant p} d(1, x_{i_j})$, we therefore have $d(x, N_G(A)) \leqslant d(abx_{i_j}, ab) \leqslant R$, as required. If A is abelian, then $ab \in C_G(A)$, so $x = abx_{i_j} \in C_G(A) \cdot x_{i_j}$, and we get $d(x, C_G(A)) \leqslant R$, as claimed. ■

On the other hand:

Claim 2. There exists $R' < \infty$ such that $N_G(A) \subseteq \mathcal{N}_{R'}(\mathcal{M}_A)$.

Proof of Claim 2. Let $x \in \mathcal{M}_A$ and let $g \in N_G(A)$. Then $g \cdot \phi_x(A)$ is an (r,r)–quasiflat contained in the (k,k)–EHQC, (k,k)–quasiflat gF_x, which is invariant under $gAg^{-1} = A$. Moreover, $F_x \subseteq X(A)$ since, by definition, $\mathcal{M}_A \subseteq X(A)$. Since $X(A)$ is $N_G(A)$–invariant, $gF_x \subseteq X(A)$. Hence, by Theorem 6.1.(5), gF_x, and in particular gx, is contained in $\mathcal{N}_{\omega_1(k)}(\mathcal{M}_A)$. We have just shown that $N_G(A) \cdot x \subseteq \mathcal{N}_{\omega_1(k)}(\mathcal{M}_A)$. Hence $N_G(A) \subseteq \mathcal{N}_{R'}(\mathcal{M}_A)$, where $R' = \omega_1(k) + d(1, \mathcal{M}_A)$. ■

The preceding two claims show that $N_G(A)$ lies at finite Hausdorff distance from $\mathcal{M}_A$. Since $\mathcal{M}_A$ is EHQC, it follows that $N_G(A)$ is also EHQC, as claimed.

Let $A_0 \leqslant A$ be the finite-index free abelian subgroup obtained from Theorem 6.1. Claim 1 implies that $\mathcal{M}_{A_0} \subseteq \mathcal{N}_R(C_G(A_0))$ for some $R < \infty$. Theorem 6.1.(4) says that $\mathcal{M}_{A_0}$ coarsely coincides with $X(A_0)$, so $N_G(A_0)$ acts on $X(A_0)$ coboundedly, and the induced action of $C_G(A_0)$ is also cobounded. This proves that $[N_G(A_0) : C_G(A_0)] < \infty$.

Let χ_0 denote the maximal r such that there is a subset $\{U_1, \ldots, U_r\} \subseteq \mathfrak{S}$ for which $U_i \perp U_j$ for $i \neq j$, and $\mathcal{C}U_i$ is unbounded for $i \neq j$. (So, χ_0 is the *rank* of $(G, \mathfrak{S})$ in the sense of [BHS21, Defn. 1.10].)

Let $A_1 \leqslant A$ be any free abelian subgroup that is contained in every subgroup of A of index at most $\chi_0!$. Then by cofiniteness of the G–action on $\mathfrak{S}$, and Proposition 4.1.(5,6), and Proposition 5.6, there is an A_1–invariant hierarchically quasiconvex subset Θ of X that lies in a neighbourhood of $X(A_1)$. Since $C_G(A_1)$ acts on $X(A_1)$ coboundedly, as shown above, it follows that $C_G(A_1)$ acts on Θ coboundedly, and is therefore hierarchically quasiconvex with respect to the HHG structure $(G, \mathfrak{S})$.

By [BHS21, Thm. 1.15] and coboundedness of the G–action, χ_0 coincides with the maximal r so that there is a quasi-isometric embedding $\mathbb{R}^r \to G$, and therefore depends only on (the QI type of) G. Since Lemma 7.2 bounds the index to which one must pass to find a free abelian subgroup of A, we can therefore find the required A_1 with $[A : A_1]$ bounded in terms of G only. In particular, since the choice of A_1 in A was made independently of the HHG structure, $C_G(A_1)$ is HQC with respect to any HHG structure. This completes the proof of Corollary 7.6.

Corollary 1.7 follows by a very similar argument. Indeed, if no element of $\mathfrak{S}$ is orthogonal to its G–translates, and A is a free abelian subgroup, then the above application of Proposition 4.1 and Proposition 5.6 allows us to take $A_1 = A$ and conclude that $C_G(A)$ is hierarchically quasiconvex. □

Next we analyse commensurators; the following result is analogous to the statement for CAT(0) and cubical groups in [HP20]. In fact, [HP20, Prop. 9.1] shows that this statement is sharp, in the sense that the finitely generated subgroup K cannot be replaced, in general HHGs, with the entire commensurator.

Corollary 7.7. *Let $(G, \mathfrak{S})$ be an HHG and let $A \leqslant G$ be a virtually abelian subgroup. Let $\mathrm{Comm}_G(A)$ be the commensurator of A in G. Let $K \leqslant \mathrm{Comm}_G(A)$ be a finitely generated subgroup. Then $K \leqslant N_G(A')$, where $A' \leqslant A$ is a finite-index subgroup.*

Proof. Let $C = \mathrm{Comm}_G(A)$. Let $K \leqslant C$ be a finitely generated subgroup. Without loss of generality, $A \leqslant K$.

Lemma 5.7.(1) states that $C\cdot\mathrm{supp}(p^{\pm}) = \mathrm{supp}(p^{\pm})$ (where $p^{\pm}$ come from applying Lemma 5.2 to A) and there is a finite-index subgroup $C' \leqslant C$ such that C' fixes each $U \in \mathrm{supp}(p^{\pm})$ and each $\pi_U(p^{\pm})$, and hence C' acts by isometries on $S = \prod_{U\in\mathrm{supp}(p^{\pm})} \mathrm{hull}_U(p^-,p^+)$, preserving the factors. Lemma 5.7.(2) gives a C–action on $(\breve{P}, \mathfrak{S} - \mathfrak{U})$ making $c : P \to \breve{P}$ a C–equivariant map (recall Definitions 5.3 and 5.5). By replacing $\breve{P}$ with its injective hull, we can assume that $\breve{P}$ is injective.[5]

Let $K' = C' \cap K$ and note that $[K : K'] < \infty$, so K' is generated by elements $k_1,\dots,k_s \in K'$. Let $A' \leqslant A \cap K'$ be a finite index free abelian subgroup, and let $A'' = \bigcap_{i=1}^{s} A' \cap k_iA'k_i^{-1}$. Let $A_i' = k_iA'k_i^{-1}$. Then each A_i' has a nonempty fixed point set $Y(A_i')$ in the injective space $\breve{P}$, and A'' has a nonempty fixed point set $Y(A'')$. Note that $Y(A_i') \subseteq Y(A'')$ for all i.

The map $\delta := c \times \prod_{U\in\mathrm{supp}(p^{\pm})} \pi_U : X(A) \to \breve{P} \times S$ is C'–equivariant, where the codomain is equipped with the diagonal action. Moreover, exactly as in the proof of Claim 1 of the proof of Theorem 6.5, there exists r, depending on A and the HHS parameters but independent of K and the choice of A' and the k_i, such that δ is an (r,r)–quasi-isometric embedding.

Now let $i \leqslant s$ and let $a \in A''$. Then $k_iak_i^{-1} \in A_i'$ since $a \in A'' \leqslant A'$. On the other hand, $a \in A'' \leqslant A_i'$. So, $[a,k_i] \in A_i'$, and hence $[a,k_i]$ fixes $Y(A_i')$ pointwise. Meanwhile, $[a,k_i]$ displaces all points in S by a uniformly bounded amount, because a and k_i both stabilise each quasiline $\mathrm{hull}_U(p^-,p^+)$ and fix the endpoints.

This shows that there exists $r' < \infty$ depending only on r and the HHS constants, and $x_0 \in G$ depending on K and the choice of $k_1,\dots,k_s$ but not on A', such that for all $a \in A''$ and all k_i, we have $d_G([a,k_i]\cdot x_0, x_0) \leqslant r'$, and $[a,k_i] \in A_i'$. Now, since A is residually finite, we could have chosen A' so that every nontrivial element of A' moves every point in G a distance more than r'. Since A_i' is conjugate in G to A', the same is true of A_i'. With this choice of A', it then follows that $[a,k_i] = 1$ for all $i \leqslant s$ and all $a \in A''$. Hence $A'' \cap K'$ is central in K'. But $A''' := A'' \cap K'$ has finite index in A since we are assuming that $A \leqslant K$.

So, A''' is a finite index subgroup of A that is central in K'. Let $\{\ell_1,\dots,\ell_m\} \subseteq K$ be a left transversal for K' and let $A_1 = \bigcap_{i=1}^{m} \ell_iA'''\ell_i^{-1}$. Now, $\ell_i \in C$, so $[A : A_1] < \infty$, and A_1 is central in K', so from the definition, it follows that A_1 is normal in K, so we are done. ☐

7.2. **Ruling out HHG structures using highest virtually abelian subgroups.** Highest virtually abelian subgroups can be used to rule out the existence of HHG structures on a group using the following corollary.

Corollary 7.8. *Let $(G,\mathfrak{S})$ be an HHG. Every highest virtually abelian subgroup $A \leqslant G$ is hierarchically quasiconvex. Thus every abelian subgroup is virtually contained in a hierarchically quasiconvex abelian subgroup.*

Proof. Suppose that A is highest. Corollary 7.6 provides a finite-index subgroup $A_1 \leqslant A$ such that $C_G(A_1)$ is hierarchically quasiconvex. If $C_G(A_1)$ has a nonabelian free subgroup, then there exists $g \in C_G(A_1)$ such that $\langle g, A_1\rangle \cong \mathbb{Z}\times A_1$, contradicting that A is highest. Hence, by [DHS20, Thm. 4.1], every finitely generated subgroup of $C_G(A_1)$ is virtually abelian, so Corollary 7.3 implies $[C_G(A_1) : A_1] < \infty$, so A_1 (and thus A) is HQC. ☐

Using this, we can describe the possible crystallographic groups that can appear as highest virtually abelian subgroups of HHGs, by generalising [PS23, Thm. 4.4]. Recall from Definition 2.53 the notion of a hyperoctahedral virtually abelian group.

[5]When replacing $\breve{P}$ by the injective hull, we keep the same HHS structure, except we modify the projections $\pi_U : \breve{P} \to \mathcal{C}U$ using a fixed quasi-isometry $\mathrm{Env}(\breve{P}) \to \breve{P}$, as in [BHS19, Prop. 1.10]. If desired, although it is not strictly necessary in this proof, we can further modify these maps as in [DHS20, Sec. 2.2] so that the action is still by HHS automorphisms.

Corollary 7.9. *Let $(G, \mathfrak{S})$ be an HHG. Let $A \leqslant G$ be a highest virtually abelian subgroup. Then A is hyperoctahedral.*

Proof. By Corollary 7.8, A is hierarchically quasiconvex in G. Hence A acts properly and coboundedly on an HHS, by Lemma 2.28, and is therefore coarsely injective, by Corollary 2.45. This, together with [Hod23, Thm. A] implies that the point group is conjugate in $GL_n(\mathbb{R})$ into the isometry group of $(\mathbb{E}^n, \|\cdot\|_\infty)$, which is to say $O_n(\mathbb{Z})$. $\square$

Remark 7.10. One can use highest virtually abelian subgroups A of a group G as an obstruction to cocompact cubulation of G, as follows: if G is cocompactly cubulated, then A is also cocompactly cubulated, by [WW17, Thm. 3.6], so A is hyperoctahedral by [Hag14] or [Hod23]. Corollary 7.9 can be used in an analogous way to rule out the existence of HHG structures in various examples. We now illustrate this in the case of Coxeter groups. ✽

Let (W, S) be a Coxeter system. Following [Kra09, Sec. 6], a *standard abelian subgroup* $A \leqslant W$ has the form $\prod_{i=1}^k H_i$, where each H_i is defined as follows: there are irreducible, non-spherical subsets $I_1, \dots, I_k$ of S such that $\langle W_{I_i}, W_{I_j} \rangle_W \cong W_{I_i} \times W_{I_j}$ for $i \neq j$, and, if I_i is affine, then $H_i \leqslant_{f.i.} W_{I_i}$ is the translation subgroup, and otherwise $H_i \cong \mathbb{Z}$. A *standard virtually abelian subgroup* has the form $\prod_i \widehat{H}_i$, where $\widehat{H}_i = W_{I_i}$ for I_i irreducible affine, and $\widehat{H}_i = H_i \cong \mathbb{Z}$ otherwise.

Corollary 7.11. *Let (W, S) be a Coxeter system. If W is a hierarchically hyperbolic group, then every maximal standard virtually abelian subgroup is hyperoctahedral.*

Proof. By [Kra09, Thm. 6.8.3], every abelian subgroup of W is virtually contained in a conjugate of a standard abelian subgroup. Hence maximal standard virtually abelian subgroups are highest virtually abelian subgroups, and the claim follows from Corollary 7.9. $\square$

For example, if W contains the $(3, 3, 3)$ triangle group as a maximal standard virtually abelian subgroup, then W is not an HHG.

7.3. **Quasiflat closing.** We now turn to the matter of the existence of higher rank free abelian subgroups in non-hyperbolic HHGs.

Lemma 7.12. *Let $(G, \mathfrak{S})$ be a normalised HHG and let $U \in \mathfrak{S}$ be such that $\mathcal{C}U$ is unbounded and $\mathcal{C}V$ is bounded for any $V \subsetneq U$. Then $\mathrm{Stab}_G(U)$ acts cocompactly on P_U and coboundedly on $\mathcal{C}U$.*

The above lemma is of independent interest for the theory of HHGs. The lemma does not hold for arbitrary $U \in \mathfrak{S}$, as can be illustrated with a suitably-chosen HHG structure on F_2.

Proof of Lemma 7.12. Let $C \geqslant 1$ be such that every bounded domain of $\mathfrak{S}$ has diameter at most C.

Claim 1. If $I \subseteq G \cdot U$ is such that $\{P_i\}_{i \in I}$ all intersect the same ball in G, then $|I| < \infty$.

Proof of Claim 1. Suppose there is an infinite subset $I \subseteq G \cdot U$ such that $\{P_i\}_{i \in I}$ all intersect the same ball B in G.

Since $\mathfrak{S}$ has a unique $\sqsubseteq$–maximal element and $\sqsubseteq$–chains are bounded, there exists a $W \in \mathfrak{S}$ that is $\sqsubseteq$–minimal with the property that $aU \sqsubseteq W$ for infinitely many $aU \in I$. By passing to an infinite subset, we can assume that this holds for all $aU \in I$. Fix a coarse parallel copy $F_W \subseteq P_W$ (recall the definition of F_W from Section 2.9).

Since G is the 0–skeleton of a locally finite graph, up to replacing I with an infinite subset, there exists $x \in \bigcap_{i \in I} P_i$. Considering the gate of x in F_W shows that there is a ball of uniform radius in F_W that intersects each P_i, so we can assume that $x \in F_W \cap \bigcap_{i \in I} P_i$. Recall from [BHS19, Prop. 5.11] that $(F_W, \mathfrak{S}_W)$ inherits an HHS structure with uniform parameters, where $\mathfrak{S}_W = \{V \in \mathfrak{S} : V \sqsubseteq W\}$. Since the product regions P_{aU} defined intrinsically in this HHS uniformly coarsely coincide with $P_{aU} \cap F_W$, we will work in $(F_W, \mathfrak{S}_W)$ and regard each P_i as a standard product region for this HHS.

For each $i \in I$ let $x_i \in P_i$ be such that $\theta_u(2(C+2E)) \leqslant d(x, x_i) \leqslant \theta_u(2(C+2E)) + D$, where θ_u is the uniqueness function from [BHS19, Defn. 1.1.(9)] and D is a uniform constant such that all standard product regions are D–coarsely connected. Since F_W is proper, there is an infinite subset $I' \subseteq I$ such that $x_i = x_j$ for all $i, j \in I'$. Let $y := x_i$ for any $i \in I'$. By the uniqueness axiom, there exists $V \in \mathfrak{S}_W$ such that $d_V(x, y) \geqslant 2(C+2E)$ which, by the definition of C and the fact that $x, y \in P_{aU}$ for all $aU \in I'$, implies that $V \perp aU$ or $V = aU$ for all $aU \in I'$. Let $I_1 := I' - \{V\}$. By [BHS19, Defn. 1.1.(3)], there exists $W' \subsetneq W$ such that $aU \sqsubseteq W'$ for all $aU \in I_1$. Since I_1 is infinite, this contradicts the $\sqsubseteq$–minimality of W. ■

Claim 1 implies, by the "pigeonhole principle for cocompact actions" (see e.g. [HR21, Prop. 7.2])), that the action of $\mathrm{Stab}_G(P_U)$ on P_U is cocompact. Moreover, since $P_U = P_{gU}$ for any $g \in \mathrm{Stab}_G(U)$, the claim also implies that the orbit $\mathrm{Stab}_G(P_U) \cdot U$ is finite, so the index of $\mathrm{Stab}_G(U)$ in $\mathrm{Stab}_G(P_U)$ is finite, and the action of $\mathrm{Stab}_G(U)$ on P_U is also cocompact. Since π_U is coarsely surjective, and coarsely factors through the gate map $\mathfrak{g}_{P_U}$, this implies that the action of $\mathrm{Stab}_G(U)$ on $\mathcal{C}U$ is cobounded. □

Theorem 7.13. *Let $(G, \mathfrak{S})$ be an HHG. Then G is hyperbolic if and only if G contains no $\mathbb{Z}^2$ subgroups.*

Proof. First normalise $(G, \mathfrak{S})$. Then there exists $C \geqslant 1$ such that every bounded domain of $\mathfrak{S}$ has diameter at most C. Given $g \in G$, let $\mathrm{Big}(g) := \{U \in \mathfrak{S} : \pi_U(\langle g \rangle)$ has unbounded diameter$\}$, any two of whose elements are orthogonal. Recall that the action of G on $(G, \mathfrak{S})$ is hierarchically semisimple.

Suppose G is not hyperbolic. We will show that there exists $f \in G$ such that $|\mathrm{Big}(f)| \geqslant 2$. Then, by Corollary 7.8, there is a highest abelian subgroup $A \leqslant G$ which contains a finite index subgroup of $\langle f \rangle$, and A is HQC and therefore must have rank $\geqslant 2$.

By [BHS21, Cor. 2.15] and non-hyperbolicity of G, there exist $U, V \in \mathfrak{S}$ such that $U \perp V$ and $\mathcal{C}U, \mathcal{C}V$ are unbounded. Using [BHS19, Def. 1.1.(3),(5)], we can assume that, for any $U' \subsetneq U$ and $V' \subsetneq V$, the domains $\mathcal{C}U', \mathcal{C}V'$ are bounded. Therefore Lemma 7.12 implies that there exist $g \in \mathrm{Stab}_G(U)$ and $h \in \mathrm{Stab}_G(V)$ such that $U \in \mathrm{Big}(g)$ and $V \in \mathrm{Big}(h)$.

If either $|\mathrm{Big}(g)| > 1$ or $|\mathrm{Big}(h)| > 1$ then we are done, so suppose $\mathrm{Big}(g) = \{U\}$ and $\mathrm{Big}(h) = \{V\}$.

Fix a point $x \in G$. Let $L := H_\theta(\langle g \rangle \cdot x)$ and $M := H_\theta(\langle h \rangle \cdot x)$. Note that L is g–invariant and M is h–invariant. Let $H := H_\theta(L \cup M)$.

Lemma 6.6 in [DHS17] provides a constant B such that $\mathsf{diam}(\pi_W(L)) \leqslant B$ for all $W \neq U$ and $\mathsf{diam}(\pi_W(M)) \leqslant B$ for all $W \neq V$. If $W \in \mathfrak{S} - \{U, V\}$, then $\pi_W(H) \subseteq \mathcal{N}_B(\pi_W(x))$, so $\mathsf{diam}(\pi_W(H)) \leqslant 3B$. Therefore, if $x, y \in H$, then $\mathrm{Rel}_{4B}(x, y) \subseteq \{U, V\}$ and, if $n \in \mathbb{Z}$ and $x, y \in g^n H$, then $\mathrm{Rel}_{4B}(x, y) \subseteq \{U, g^n V\}$. If $x, y \in H \cap g^n H$ and $g^n V \neq V$, then $\mathrm{Rel}_{4B}(x, y) \subseteq \{U\}$. Since $L \subseteq H \cap g^n H$, this, together with the uniqueness axiom and the definition of H, implies that $H \cap g^n H \subseteq \mathcal{N}_D(L)$, for some constant $D \geqslant 1$ which is independent of n.

Fix a point $x_0 \in L$. There exists $k \geqslant 2$ such that, for each $n \in \mathbb{N}$, there exists $x_n \in g^n H$ such that $d(x_0, x_n) \leqslant kD$ and $d(x_n, L) > D$. The space G is locally finite, so there exists $n \neq m$ such that $x_n = x_m$. But then $g^{-n} x_n \in H \cap g^{m-n} H$ and $d(L, g^{-n} x_n) = d(L, x_n) > D$, so we have $g^{m-n} V = V$. By a symmetric argument, there is a non-zero power of h which fixes U. It follows by [Gen19, Prop. 6.68] that there exists $f \in \mathrm{Stab}_G(U) \cap \mathrm{Stab}_G(V)$ such that $\{U, V\} \subseteq \mathrm{Big}(f)$. □

Recall that the *rank* of a hierarchically hyperbolic group $(G, \mathfrak{S})$ is the maximal k such that $\mathfrak{S}$ contains a set of k pairwise orthogonal elements $U_1, \ldots, U_k$ such that $\mathcal{C}U_i$ is unbounded for all $i \leqslant k$. Theorem 7.13 generalises as follows:

Theorem 7.14. *Let $(G, \mathfrak{S})$ be an HHG with rank ν. Then there exists $\mathbb{Z}^\nu \leqslant G$.*

Proof. First normalise $(G, \mathfrak{S})$. Then there exists $C \geqslant 1$ such that every bounded domain of $\mathfrak{S}$ has diameter at most C. Given $g \in G$, let $\mathrm{Big}(g) := \{U \in \mathfrak{S} : \pi_U(\langle g \rangle)$ has unbounded diameter$\}$. Recall that the action of G on itself is hierarchically semisimple.

Claim 1. Suppose $\mathfrak{V} \subseteq \mathfrak{S}$ is a set of k pairwise orthogonal elements such that, for each $V \in \mathfrak{V}$, the domain $\mathcal{C}V$ is unbounded and $\mathcal{C}V'$ is bounded for all $V' \subsetneq V$. Then there exists $g \in G$ such that $\mathfrak{V} \subseteq \mathrm{Big}(g)$.

Proof of Claim 1. Lemma 7.12 implies that $\mathrm{Stab}_G(V)$ acts coboundedly on $\mathcal{C}V$ for each $V \in \mathfrak{V}$ which, since the action is semisimple, implies that there is a $\mathcal{C}V$–loxodromic element in each $\mathrm{Stab}_G(V)$. If $k = 1$ then we are done, so suppose $k > 1$ and the claim holds for sets of $k - 1$ elements.

Fix $W \in \mathfrak{V}$ and let $\mathfrak{V}' := V - \{W\}$. Let $g, h \in G$ be such that $W \in \mathrm{Big}(g)$ and $\mathfrak{V}' \subseteq \mathrm{Big}(h)$. Fix a point $x \in G$ and let $L_g := H_\theta(\langle g \rangle \cdot x)$, $L_h := H_\theta(\langle h \rangle \cdot x)$. Lemma 6.6 in [DHS17] provides a constant $B > 0$ such that $\mathsf{diam}(\pi_U(L_g)) \leqslant B$ for all $U \in \mathfrak{S} - \mathrm{Big}(g)$ and $\mathsf{diam}(\pi_U(L_h)) \leqslant B$ for all $U \in \mathfrak{S} - \mathrm{Big}(h)$. Note that L_g and L_h are g and h–invariant respectively. Let $H := H_\theta(L_g \cup L_h)$.

Let $\mathfrak{U} \subseteq \mathrm{Big}(g) \cup \mathrm{Big}(h)$ be the set of all $U \in \mathrm{Big}(g) \cup \mathrm{Big}(h)$ such that some non-zero power of g stabilises U. Note that $\mathrm{Big}(g) \subseteq \mathfrak{U}$. Let

$$M := \{x \in H : d_V(x, \pi_V(L_g)) \leqslant E \;\forall\; V \in \mathrm{Big}(h) - \mathfrak{U}\}$$

and note that M is g–invariant and HQC. Replace g by positive power of itself, so that $g \in \bigcap_{U \in \mathfrak{U}} \mathrm{Stab}_G(U)$ and, if $gU \in \mathrm{Big}(g) \cup \mathrm{Big}(h)$ then $gU = U$.

If $U \notin \mathrm{Big}(g) \cup \mathrm{Big}(h)$, then $\pi_U(H) \subseteq \mathcal{N}_B(\pi_U(x))$, so $\mathsf{diam}(\pi_U(H)) \leqslant 3B$. Therefore, if $x, y \in H$, then $\mathrm{Rel}_{4B}(x, y) \subseteq \mathrm{Big}(g) \cup \mathrm{Big}(h)$. It follows that, if there exists $V \in \mathrm{Big}(h) - \mathfrak{U}$, then $V \notin \mathrm{Rel}_{4B}(x, y)$ for any $x, y \in g^n H$ and $n \in \mathbb{N}$. In other words, for any $n \in \mathbb{N}$ and $x, y \in H \cap g^n H$, we have $\mathrm{Rel}_{4B}(x, y) \subseteq \mathfrak{U}$. This implies, by the uniqueness axiom, that $H \cap g^n H \subseteq \mathcal{N}_D(M)$, where $D \geqslant 0$ is a constant independent of n.

Fix a point $x_0 \in \mathfrak{g}_M(L_h)$. If $\mathrm{Big}(h) - \mathfrak{U} \neq \varnothing$, then there exists $r \geqslant 2$ such that, for each $n \in \mathbb{N}$, there exists $x_n \in g^n H$ such that $d(x_0, x_n) \leqslant rD$ and $d(x_n, M) > D$. (More precisely, start with $\mathfrak{g}_{L_h}(x_0)$, and use realisation to move to a new point whose projection to $\mathcal{C}V$ only differs by more than a uniformly bounded amount from that of $\mathfrak{g}_{L_h}(x_0)$ when $V \in \mathrm{Big}(h) - \mathfrak{U}$. Then apply g^n to the new point.) The space G is locally finite, so there exists $n \neq m$ such that $x_n = x_m$. But then $g^{-n} x_n \in H \cap g^{m-n} H$ and $d(M, g^{-n} x_n) = d(M, x_n) > D$, which is a contradiction. Thus $\mathfrak{U} = \mathrm{Big}(g) \cup \mathrm{Big}(h)$.

By a symmetric argument, there is a non-zero power of h which fixes each $U \in \mathrm{Big}(g) \cup \mathrm{Big}(h)$. It follows by [Gen19, Prop. 6.68] that there exists $f \in \bigcap_{V \in \mathfrak{V}} \mathrm{Stab}_G(V)$ such that $\mathfrak{V} \subseteq \mathrm{Big}(f)$. ■

By definition, there exists a set $\mathfrak{V} \subseteq \mathfrak{S}$ of ν pairwise orthogonal elements such that $\mathcal{C}V$ is unbounded for all $V \in \mathfrak{V}$. Using [BHS19, Def. 1.1.(3)], we can assume without loss of generality that, if $V \in \mathfrak{V}$ and $V' \subsetneq V$, then $\mathcal{C}V'$ is bounded. By Claim 1, there exists $g \in G$ such that $\mathrm{Big}(g) = \mathfrak{V}$ and, by Corollary 7.8, any highest abelian subgroup $A \leqslant G$ virtually containing $\langle g \rangle$ is HQC and therefore has rank ν. □

Appendix A. Chasing constants

Throguhout, we used some results from the literature on hierarchical hyperbolicity that take as input an HHS $(X, \mathfrak{S})$ and yield some conclusion involving various quantities depending only on the HHS parameters for $(X, \mathfrak{S})$. This uniformity of constants is not always explicitly asserted in the versions of those statements in the literature, but can be extracted from their proofs. In this appendix, we make the dependence of the constants explicit in these statements.

A.1. **Strong distance formula.** The definition of an HHS is designed to enable one to prove several results that then serve as the main tools; most notable is the *distance formula* from [BHS19], of which special cases appear in [MM00, Sis13, BHS17a]. In the statement, we use *threshold* notation: given $A, T \in \mathbb{R}$, let $\{\!\{A\}\!\}_T := A$ if $A \geqslant T$ and 0 if $A < T$.

Theorem A.1 (Strong distance formula)**.** *For all $\lambda \geqslant 1$, the following holds. Let $(X, \mathfrak{S})$ be a hierarchically hyperbolic space and let $h : [0, \infty) \to [0, \infty)$ be such that $\lambda^{-1}r - \lambda \leqslant h(r) \leqslant \lambda r + \lambda$ for all $r \geqslant 0$. Then there exists $T_0 \geqslant 2\lambda$, depending only on λ and the HHS parameters, such that for all $T \geqslant T_0$, there exists $\kappa \geqslant 0$, depending only on T, λ, and the HHS parameters, such that*

$$\kappa^{-1} d_X(x, y) - \kappa \leqslant \sum_{U \in \mathfrak{S}} \{\!\{h(d_U(x, y))\}\!\}_T \leqslant \kappa d(x, y) + \kappa$$

for all $x, y \in X$.

Proof. The case $\lambda = 1$ is the original distance formula for HHSes, which is [BHS19, Thm. 4.5], whose proof makes clear that the threshold T_0 for $\lambda = 1$ depends only on the HHS parameters and, for any $T \geqslant T_0$, one obtains κ depending only on T and the HHS parameters. The general case follows from the case $\lambda = 1$ by the exact same argument as is used in the proof of [BHMS24b, Thm. 2.9]; in fact, the latter theorem itself implies the present theorem as soon as we observe that the proof works for any T chosen sufficiently large in terms of λ and the "threshold" T_0 from the $\lambda = 1$ case. □

The interested reader can refer to [CRHK24, Rem. 12.10], which is about an alternate proof due to Durham that also yields uniform constants.

A.2. **Cubical approximation theorem.** Next, we recall the cubical approximation theorem from [BHS21], of which there are several variants with different proofs, for instance in [DMS23, Dur23, DZ22, CRHK24]. The following version is the same as the statements from [BHS21] and [CRHK24], except it is explicit about what the constants depend on.

Proposition A.2 (Cubical approximation)**.** *Let $(X, \mathfrak{S})$ be a hierarchically hyperbolic space of complexity χ. Then there exists M_0, depending on the HHS parameters only, such that for all $k \in \mathbb{N}$ there exists C, depending only on the HHS parameters and k, such that the following holds.*

Let $M_1 := 100 M_0 C k$. Then for all $M \geqslant M_1$, there is a constant $Q \geqslant 1$, depending on M, k, and the HHS parameters only, satisfying the following.

Let $A = \{x_1, \ldots, x_k\} \subset X$ and let $\mathfrak{U}$ be the set of $U \in \mathfrak{S}$ such that $d_U(x_i, x_j) \geqslant M$ for some i, j. Then there exists a CAT(0) cube complex $\square_A$ and a map $f : \square_A \to X$ such that:

- *f is a (Q, Q)–quasi-isometric embedding and $d_{Haus}(f(\square_A), H_\theta(A)) \leqslant Q$.*
- *There exist $\hat{x}_{i_1}, \ldots, \hat{x}_{i_s} \in \square_A$ such that $d_X(f(\hat{x}_{i_j}), x_j) \leqslant C$ for all j and $\square_A$ is equal to the convex hull in $\square_A$ of $\{\hat{x}_{i_1}, \ldots, \hat{x}_{i_s}\}$. In particular, $\square_A$ has finitely many cubes.*
- *The map f is Q–quasimedian.*
- *Each hyperplane h of $\square_A$ is labelled by an element of $\mathfrak{U} \subset \mathfrak{S}$, and hyperplanes h, h' cross if and only if their labels U, U' satisfy $U \perp U'$. In particular, $\dim \square_A \leqslant \chi$.*

Finally, if γ is a combinatorial geodesic in $\square_A$, then $f \circ \gamma$ is a (D, D)–hierarchy path in X, where D depends only on M, k, and the HHS parameters.

Proof. The only difference between this proposition and [CRHK24, Prop. 18.1] is that the current proposition says explicitly what the constants M_0 and Q depend on. As noted in [CRHK24, Sec. 18], everything in the statement (except for the claim about the dependency of constants) is given by [BHS21, Thm. 2.1], except for the final assertion about hierarchy paths. Therefore, we will first explain why the proof of [BHS21, Thm. 2.1] gives effective constants, i.e., why M_0 can be chosen in terms of the HHS parameters, and Q can be chosen in terms of k and the HHS parameters. Then we will verify the claim about hierarchy paths.

Constants in the proof from [BHS21]. Now we trace through the proof of [BHS21, Thm. 2.1], starting with the constructions in [BHS21, Sec. 2] before the theorem:

(1) The finite trees T_U are $(1, C)$–quasi-isometrically embedded in $\mathcal{C}U$, where $C \geqslant 1$ is a constant depending on E and k.
(2) The constant M in [BHS21] is what we have here called M_0C. In [BHS21], the set $\mathfrak{U}$ is chosen to consist of those $U \in \mathfrak{S}$ for which $\pi_U(A)$ has diameter at least $100M_0Ck$, in our notation. So, once we have chosen M_0, the constant M_1 from our current statement is determined. The constant M_0 is chosen by imposing various (finitely many) constraints as the proof proceeds, all in terms of the HHS parameters only; we will mention these as they arise. The first is just that $M_0 > E$, and hence $M > E$.
(3) The cardinality of $\mathfrak{U}$ is bounded in terms of $\mathsf{diam}_X(A)$ and the HHS parameters only (including χ), using the large link axiom, although what's needed is just that $\mathfrak{U}$ is finite. (See the remark on [BHS21, p. 947], or [CRHK24, Lem. 13.5].)
(4) The only constraints on constants arising in the construction of the walls in $H_\theta(A)$ (and hence the dual cube complex $\square_A$, which is called $\mathcal{Y}$ in [BHS21]) is that $M_0 \geqslant E$.
(5) In Section 2.2 of [BHS21], there are three lemmas, Lemmas 2.3–2.5, whose hypotheses require that (in our notation) $M_0 \geqslant 10E$, which is the most restrictive hypothesis in any of these lemmas.
(6) Lemma 2.6 introduces a constant $\tau = \tau(M, k)$. The proof of this lemma makes it explicit that it is sufficient to choose $\tau = 50Mk(k-2)$.
(7) Lemma 2.7 introduces a constant η; the statement says that this depends only on M, k, and the HHS X, but the dependency on X is only via the HHS parameters; specifically, the proof shows that $\eta = 100(Mk + C + 2E)$ works.
(8) In the proof of Theorem 2.1 itself, the choice of η determines another constant ξ, which depends only on η and the HHS parameters; specifically, $\xi = \eta r_0$, where r_0 is the realisation theorem constant (Theorem 2.15) discussed above.
(9) Lemma 2.8 does not influence the constants, and the constant N from Lemma 2.9 depends only on an HHS parameter, the complexity, via Ramsey's theorem.
(10) Lemma 2.10 provides a constant T depending on M, k, ξ, τ and the HHS $(X, \mathfrak{S})$. The proof of Lemma 2.10 again makes it clear that the dependence on $(X, \mathfrak{S})$ is only via dependence on the HHS parameters (not on the specific choice of space X etc.). This uses [BHS21, Lem. 1.6], the "passing up large projections" lemma, in which the output N_0 depends on the input and on the HHS parameters (in particular, the constants in the large link axiom and the complexity) only.
(11) Lemma 2.12 does not involve constants. The proof of Lemma 2.11 makes clear that the constant implicit in the word "coarsely" in the statement depends on E and θ and hence only on the HHS parameters.
(12) Lemma 2.13 provides a quasimedian constant. One must read the proof to check that this depends only on the HHS parameters and k, but the key facts are that

the constants τ and η depend only on those things, and that the distance bounds involved in the construction of the coarse median map μ (see Section 2.5 above) also depend only on the HHS parameters.

(13) Finally, we examine the proof of Theorem 2.1 itself. The constants listed just above appear in that proof, and are bounded in terms of the HHS parameters and M, k; hence the same is true of any quantity computed from those constants via some procedure that only depends on the HHS parameters. In particular, the constant C_1' in the proof of Theorem 2.1 is obtained by evaluating the uniqueness function, and the hierarchical quasiconvexity function h, on uniform inputs, and is thus uniform.

The constant C_1'' is a distance formula quasi-isometry constant, with uniform threshold (depending on E and M), which is therefore uniform (see Theorem A.1). The constant C_1''' depends in a uniform way on the output of Lemma 2.6, i.e. on τ, and is therefore uniform. Finally, the constant C_1'''' is the uniform constant from Lemma 2.13.

This establishes that M_0 depends only on the HHS parameters, C depends only on the HHS parameters and k, and Q depends on the HHS parameters, k, and M.

It just remains to prove the statement about hierarchy paths. As noted in [CRHK24, Rem. 18.2], this follows from the fact that f is a (Q, Q)–quasimedian quasi-isometric embedding together with Proposition 2.25. That D depends only on M, k, and the HHS parameters follows from the latter proposition and the fact, established above, that Q is uniform. ☐

Proposition A.2 immediately yields the following (which is also proven in [BHS19, Sec. 7] without using cubical approximation):

Corollary A.3. *Let $(X, \mathfrak{S})$ be an HHS. Let $\mu : X^3 \to 2^X$ be the coarse map from Construction 2.17. Then μ is a coarse median in the sense of Definition 2.16, and the coarse median parameters depend only on the HHS parameters.*

A.3. **Non-canonical cone-off.** The following proposition is almost exactly the same as Proposition 19.1 in [CRHK24] except for the clause about HHS parameters.

Proposition A.4 (Equivariant cone-off, with parameters)**.** *Let $(X, \mathfrak{S})$ be an HHS, let $A \leqslant \mathrm{Isom}(X)$ be a group of HHS automorphisms, and let $\mathfrak{U} \subseteq \mathfrak{S}$ be an A–invariant downward-closed set. Let $(\widehat{X}, \widehat{d})$ be the non-canonical cone-off from Definition 2.38. Then the pair $(\widehat{X}, \mathfrak{S} - \mathfrak{U})$ is a hierarchically hyperbolic space, whose HHS parameters are bounded above in terms of the HHS parameters of $(X, \mathfrak{S})$ only. Moreover, the A–action on $(X, \mathfrak{S})$ induces an action of A on $(\widehat{X}, \mathfrak{S} - \mathfrak{U})$ by HHS automorphisms for which the A–action on $(X, \widehat{d})$ is an action by (C, C)–quasi-isometries, where C depends only on the HHS parameters of $(X, \mathfrak{S})$.*

Proof. This is [CRHK24, Prop. 19.1] (in turn a variant of [BHS17b, Prop. 2.4]) except we need to justify the statements about the HHS parameters of $(\widehat{X}, \mathfrak{S} - \mathfrak{U})$ and C. As noted in the proof of [CRHK24, Prop. 19.1], the A–action is by quasi-isometries, by the distance formula for the HHS $(\widehat{X}, \mathfrak{S} - \mathfrak{U})$. Hence C depends only on the HHS parameters of $(\widehat{X}, \mathfrak{S} - \mathfrak{U})$, so it suffices to prove the latter just depend on the HHS parameters for $(X, \mathfrak{S})$.

This follows by examining the proof that $(\widehat{X}, \mathfrak{S} - \mathfrak{U})$ is in HHS. In [BHS17b, Sec. 2], the proof of Proposition 2.4 shows that all of the "new" HHS parameters except for the uniqueness function are bounded in terms of the "old" ones by a straightforward observation. For example, the underlying set of $\widehat{X}$ is just X, and for $U \in \mathfrak{S} - \mathfrak{U}$, the projection $X \to \mathcal{C}U$ is the same coarse map π_U as in the original HHS structure. So the consistency, bounded geodesic image, large links, and partial realisation axioms, which do not mention the metric on X, hold unchanged for the new HHS structure. The same is true of the axioms that just concern the relations $\sqsubseteq, \perp, \pitchfork$ on $\mathfrak{S}$ (such as finite complexity). The first place where the difference between d and $\widehat{d}$ matters is in the proof that $\pi_U : \widehat{X} \to \mathcal{C}U$ is coarsely lipschitz, but the constant is computed in [BHS17b] in terms of the HHS parameters of $(X, \mathfrak{S})$.

The remaining, and most complicated, thing to check is that the uniqueness function for $(\widehat{X}, \mathfrak{S} - \mathfrak{U})$ is bounded above in terms of the HHS parameters for $(X, \mathfrak{S})$. Here we follow the proof in [CRHK24, Proposition 19.1], which uses Proposition A.2 and yields explicit estimates. Tracing through that proof, the inputs are:

- The uniqueness function θ_u for $(X, \mathfrak{S})$.
- The constant θ, which depends only on $(X, \mathfrak{S})$, given by [BHS19, Prop. 6.15].
- The HHS constant E.
- Constants M_0, M_1, M that depend only on the HHS parameters of $(X, \mathfrak{S})$ and are provided by Proposition A.2, with $k = 2$.
- A constant $N = N(M)$ which depends only on M and the HHS parameters, via [CRHK24, Proposition 13.1], essentially a consequence of the large link and finite complexity axioms for $(X, \mathfrak{S})$.

In the remainder of the proof, we are given an arbitrary $\kappa \geqslant 0$ and points $x, y \in \widehat{X}$ with $d_U(x, y) \leqslant \kappa$ for all $U \in \mathfrak{S} - \mathfrak{U}$. The goal is to bound $\widehat{d}(x, y)$ in terms of κ and the HHS parameters of $(X, \mathfrak{S})$. First, Claim 12 in the proof of [CRHK24, Proposition 19.1] produces a constant ν that depends only on κ and the HHS parameters of $(X, \mathfrak{S})$. Next, the constant N in Claim 13 is the one mentioned above. In the final part of the proof, induction on the complexity of the original HHS yields a constant ν^1, explicitly stated to depend only on the original HHS parameters and κ, and the estimate $\widehat{d}(x, y) \leqslant N(\nu + \nu^1)$. This shows that the new uniqueness function is bounded in terms of the old HHS parameters, as required. □

A.4. **Coarse injectivity.** We recall the following theorem from [HHP23, Theorem A]:

Theorem A.5. *Let $(X, \mathfrak{S})$ be an HHS. There is a metric σ on X and constants M_1, M_2 such that (X, σ) is M_1–coarsely injective and M_2–quasi-isometric to (X, d), and any subgroup $H \leqslant \mathrm{Isom}(X, d)$ acting by HHS automorphisms on $(X, \mathfrak{S})$ acts by isometries on (X, σ). Moreover the constants M_1, M_2 depend only on the HHS parameters.*

Proof. Using Lemma 2.47, we can assume that X is a graph with the usual combinatorial metric. The theorem is now exactly the content of [HHP23, Theorem A]; we just need to go through their proof and check that the constants depend only on the HHS parameters.

The definition of σ requires an arbitrary choice of positive constant; we choose 1.

(1) By Proposition A.2, there exists κ, depending only on the HHS parameters, such that: for any pair of points $x, y \in X$, there exists a finite CAT(0) cube complex Q with dimension at most χ and a κ–quasimedian (κ, κ)–quasi-isometry $\lambda : Q \to H_\theta(x, y)$, with θ as in Lemma 2.36. There exists $\kappa' \geqslant 1$, depending only on κ and $h_\theta(0)$, such that λ has a quasi-inverse which is a (κ', κ')–quasi-isometry and is κ'–quasi-median.

(2) By [NWZ19, Lemmas 2.18 and 2.19], there exists $H_5 \geqslant 0$, which depends only the coarse median parameters of X, and hence, by Corollary A.3, only depends on the HHS parameters, such that:

$$d(\mu(a, b, \mu(x, y, z)), \mu(\mu(a, b, x), \mu(a, b, y), z)) \leqslant H_5,$$
$$d(\mu(a, b, \mu(x, y, z)), \mu(a, b, x), \mu(a, b, y), \mu(a, b, z)) \leqslant H_5$$

for all $a, b, x, y, z \in X$.

(3) Let $L := \max\{1 + h(0), 2 + H_5\}$. Then, according to the proof of [HHP23, Proposition 2.16], we can take $M_2 = \chi {\kappa'}^2 L$.

(4) By [HHP23, Lemma 2.22], there exists $\varepsilon \geqslant 0$ such that, for any $x, y, z \in X$ such that $x = \mu(x', y, z)$ for some $x' \in X$, we have $d(x, \mu(x, y, z)) \leqslant \varepsilon$. One can choose $\varepsilon :=$

$r'_0 + r_0 + h(0)(r'_0 + 1)$, where r_0, r'_0 are constants provided by [Bow19, Lemma 8.1(2)] which only depend on the coarse median parameters of X, which in turn only depend on the HHS parameters, again by Corollary A.3.

(5) Let

$$C'_\sigma := 64(1+\kappa)\chi + 4M_2(1+\kappa) + 4\kappa + 4 + 2(4h(0)+4).$$

Then (X,σ) is C'_σ–weakly roughly geodesic [HHP23, Prop. 2.21].

(6) For $M_3 := \varepsilon + 2 + 3C'_\sigma$, balls in (X,σ) are M_3–median quasiconvex [HHP23, Lem. 2.23].

(7) There exists $k : [0,\infty) \to [0,\infty)$, depending only on M_3 and the HHS parameters, such that balls in (X,σ) are k–hierarchically quasiconvex [RST23, Prop. 5.11].

(8) For all $r \geqslant 0$, there exists $R(r) \geqslant 0$, depending only on r and E, such that the following holds. If $\mathcal{Q}$ is a collection of bounded k–hierarchically quasiconvex subspaces of X which are r–close (with respect to d), then there is a point $x \in X$ with $d(x,Q) \leqslant R(r)$ for each $Q \in \mathcal{Q}$ [HHP23, Theorem 3.5].

(9) Following the proof of [HHP23, Corollary 3.6], let us check that (X,σ) is coarsely injective with constants only depending on the HHS parameters. Let $\{B_\sigma(x_i, r_i) : i \in I\}$ be a family of balls in (X,σ) such that $\sigma(x_i,x_j) \leqslant r_i + r_j$ for all $i,j \in I$. Since (X,σ) is weakly roughly geodesic with constant C'_σ, the balls $\{B_\sigma(x_i, r_i + 2C'_\sigma) : i \in I\}$ intersect pairwise. For each $i \in I$, let B_i be the image of $B_\sigma(x_i, r_i + 2C'_\sigma)$ under the identity M_2–quasi-isometry $(X,\sigma) \to (X,d)$. Each B_i is M_3–coarsely median convex and therefore k–hierarchically quasiconvex. Thus there is a point $x \in X$ such that $d(x, B_i) \leqslant R(0)$ for each $i \in I$. So (X,σ) is coarsely injective with constant $M_1 := M_2(R(0) + C'_\sigma) + M_2$. ☐

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