RESEARCH STATEMENT

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My research is primarily concerned with geometric group theory. This area of mathematics can be described as the study of groups (often finitely generated) via their geometry, or more precisely, via their actions on metric spaces. It brings together a range of problems and techniques from different areas under the principle that groups, and the geometric objects they interact with, are best understood in tandem. It has its roots in combinatorial group theory and low dimensional topology but also has significant intersections with numerous areas of mathematics, including topological and ergodic dynamics, analysis, and model theory, to name a few.

An important aspect of the theory is that certain metric spaces provide more information about the groups which act on them than others. Take for example a simple but ground-breaking observation due to Serre in the 1970s: a group is free if and only if it admits a free action on a tree. From this, one obtains as an immediate corollary that any subgroup of a free group is free, otherwise known as the Nielson–Schreier Theorem. Serre's work takes this idea further, and fully characterises the groups which act on trees without a global fixed point in terms of Bass' graphs of groups, establishing what is now known as Bass–Serre theory and revolutionising the way that combinatorial group theory is studied and understood. Furthermore, non-trivial actions of a group G on trees imply the existence of certain unitary representations of G and play a key role in understanding the automorphism group of G.

The properties that trees possess which make them such powerful tools for understanding the groups which act on them can be interpreted in many different ways. The fact that their geodesic triangles are all tripods makes them extreme examples of Gromov's CAT(0) spaces and δ-hyperbolic spaces. This same property implies that they are median, a property which plays a key role in my research and that we will return to. The fact that any edge separates a tree into precisely two connected components makes them a motivating example of CAT(0) cube complexes, which can themselves be thought of as higher dimensional analogues of trees. Staying in one dimension but removing the need to be discrete, one obtains $\mathbb{R}$-trees, which are geodesic spaces in which every pair of points is connected by a unique path. In each of theses classes of spaces, a structure theorem classifying all the groups which act on them is considered entirely out of reach. Far from a cause for despair, this is reflective of the diversity of behaviour one witnesses among such groups and the richness of the theories which have been developed to study them. While tremendous progress has been made in the decades during which geometric group theory has developed (leading for example to the proof of Waldhausen's virtual Haken conjecture and Thurston's virtual fibering conjecture by Agol and Wise, and the solution to the Tarski problem by Kharlampovich–Myasnikov and Sela), much remains to be done and many fundamental questions have yet to be answered.

My research can be roughly divided into two overarching objectives: studying and understanding the groups which act on median spaces, and studying the subgroup structure of finitely generated groups which act on the various spaces mentioned above.

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1. Actions on median spaces

A *median space* is a metric space (X, d) such that, for any triple of points $x_0, x_1, x_2 \in X$, there exists a unique point $m \in X$, called the *median* of x_0, x_1, x_2, such that

$$d(x_i, x_j) = d(x_i, m) + d(m, x_j)$$

for all $i \neq j$. These spaces are a special case of the more general notion of a median algebra, which shows up in many different fields of research including order theory and computer science. The interest in them in geometric group theory is relatively recent but rapidly increasing due to their ability to explain many interesting phenomena. Below I describe some important examples of these spaces and some of my work on them, as well as problems I am interested in exploring.

1.1. **ℝ-trees.** Real trees (or ℝ-trees) were first defined by Tits in [Tit77]. They can be characterised as median spaces of rank 1, or as geodesic metric spaces in which every pair of points is connected by a unique simple path. Significant interest in these spaces arose following the work of Morgan–Shalen [MS84], who established connections between real trees, hyperbolic spaces and Thurston's theory of measured laminations. In a precise sense, ℝ-trees are the limits of hyperbolic spaces, or what one sees when one views hyperbolic spaces from 'infinitely far away'; precisely, the *asymptotic cone* of a hyperbolic space is an ℝ-tree. As a result, the study of ℝ-trees has proved extremely useful to the study of hyperbolic groups and their automorphism groups (see for instance [BF95; MS84; RS94; Sel95; Sel09]).

Given a group acting freely and transitively on a metric space, one can think of the group itself as being equipped with a left-invariant metric structure. If T is a simplicial tree and G is a group acting freely and vertex-transitively on T then the degree κ of T is even and G is the free group of rank $\kappa/2$. Thus such an action completely determines the algebraic structure of G. Work of Casals-Ruiz, Hagen and Kazachkov shows that this is no longer the case when one passes to ℝ-trees [CHK24]. Much of my recent work has involved building such actions on ℝ-trees, and one consequence of my results is the following, which shows the extent to which this uniqueness fails in ℝ-trees:

Corollary 1.1 ([Azu25, Corollary B])**.** *There are $2^{2^{\aleph_0}}$ pairwise non-isomorphic groups which admit a free and transitive action on the universal real tree with valence $2^{\aleph_0}$.*

The universal real tree in the above corollary is the unique complete real tree with valence $2^{\aleph_0}$ (i.e. every point in the tree disconnects it into $2^{\aleph_0}$ connected components). The existence and uniqueness of such a tree was proven by Mayer, Nikiel and Oversteegen for any cardinal $\kappa \geqslant 2$ [MNO92]. A somewhat surprising fact, given the rich landscape of free transitive actions on the universal real tree of valence $2^{\aleph_0}$, is that such actions do not exist at all for smaller valence trees:

Theorem 1.2 ([Azu25, Theorem C])**.** *Let $3 \leqslant \kappa < 2^{\aleph_0}$ be a cardinal. There are no free transitive actions on the universal ℝ-tree T_κ with valence κ.*

Corollary 1.1 is a consequence of Theorem 1.3 below. This theorem provides the first examples of free transitive actions on a real tree (or, to my knowledge, any length space) where one can find a line whose stabiliser is a dense subgroup of the additive reals. It relies on a new way of constructing groups which is inspired by the explicit constructions of universal real trees by Dyubina–Polterovich [DP01] and the construction of 'covering' real trees for any length space by Berestovskiĭ–Plaut [BP10]. As in the above constructions, the elements of the groups I construct are (equivalence classes of) maps $f : [0, \ell] \to X$, where X is a fixed set. The new idea is that the set X can be equipped with an involution and an action of $\mathbb{R}$ so that a group operation can be defined where the product of some pair (f,g) depends on the length of the domain of f. As is usually the case when defining group operations from scratch, the difficulty lies in guaranteeing associativity. For this purpose, I developed an abstract framework which takes as an input an algebraic object called an *ore*, which is a set equipped with a monoid and semi-lattice structure satisfying some axioms, and outputs a group with an explicit operation and median operator [Azu25, Section 3]. This framework provides an important technical tool for most of the results mentioned this subsection and Subsection 1.3.

Let $\mathrm{Sub}_{NC}(\mathbb{R})$ denote the set of non-cyclic subgroups of the additive group $\mathbb{R}$ and let $\mathcal{K}$ denote the set of cardinals κ with $\kappa \leqslant 2^{\aleph_0}$. Given any action $G \curvearrowright X$ and any subset $Y \subseteq X$, the (global) stabiliser of Y in G is denoted by $\mathrm{Stab}_G(Y)$.

Theorem 1.3 ([Azu25, Theorem A])**.** *Let $\iota : \mathrm{Sub}_{NC}(\mathbb{R}) \to \mathcal{K}$ be any map which is supported on $\leqslant 2^{\aleph_0}$ elements of $\mathrm{Sub}_{NC}(\mathbb{R})$. Then there exists a group G and a free transitive action of G on the universal real tree T with valence $2^{\aleph_0}$ such that the following holds. For each $H \leqslant \mathbb{R}$, let A_H be the set of orbits $G \cdot L$ such that $L \subseteq T$ is a line and the induced action of $\mathrm{Stab}_G(L)$ on L is isomorphic to the action of H on $\mathbb{R}$ by translations. If $H \leqslant \mathbb{R}$ is non-cyclic then $|A_H| = \iota(H)$.*

1.2. **Incomplete homogeneous $\mathbb{R}$-trees.** The original definition stipulates that an $\mathbb{R}$-tree must be complete. Over time however, this requirement has been dropped, mostly due to the fact that many arguments involving real trees do not require completeness. Moreover, incomplete $\mathbb{R}$-trees arise naturally in certain settings: they appear for instance as 'factors' in the $\mathbb{R}$-cubing decomposition of the asymptotic cones of hierarchically hyperbolic groups, including mapping class groups [CHK24].

Although the existence of universal (and thus unique) complete real trees with any given valence has been known for some time and is a useful feature of the theory, the situation for incomplete homogeneous real trees is not well-understood. In a current project, I am making progress on filling this gap in our understanding:

Theorem 1.4 (Work in progress)**.** *Let $\kappa > 2$ be a cardinal. There exist uncountably many pairwise non-isometric incomplete homogeneous real trees with valence κ.*

More precisely, given a cardinal $\kappa > 2$. there is an incomplete homogeneous $\mathbb{R}$-tree $T_\kappa^{[\alpha]}$ associated to each countable ordinal $\alpha \geqslant 1$. If $\alpha < \beta$, then there is a natural isometric embedding $T_\kappa^{[\alpha]} \hookrightarrow T_\kappa^{[\beta]}$, but $T_\kappa^{[\alpha]}$ is not isometric to $T_\kappa^{[\beta]}$. The complete universal real tree with valence $2^{\aleph_0}$ can be viewed as an increasing union $\cup_{\alpha<\omega_1} T_\kappa^{[\alpha]}$.

Since valence alone is not sufficient to distinguish between incomplete homogeneous real trees, classifying them presents new challenges compared to the complete case.

Problem 1.5. *Classify the homogeneous incomplete real trees.*

More precisely, it is not yet clear to me whether there should exist any incomplete homogeneous real trees outside of those mentioned above:

Question 1.6. Does there exist an incomplete homogeneous real tree T which has valence $\kappa > 2$ but is not isometric to $T_\kappa^{[\alpha]}$ for any countable ordinal $\alpha \geqslant 1$?

Another natural question is that of which homogeneous real trees admit free transitive actions. It is already clear from Theorem 1.2 that this is a much more restrictive condition than homogeneity. A partial answer is provided by the following theorem:

Theorem 1.7 ([Azu25, Theorem C]). *Let $\kappa \geqslant 3$ be any cardinal. There exists a free transitive action $G \curvearrowright S_\kappa$, where G is a group and S_κ is an incomplete $\mathbb{R}$-tree with valence κ, if and only if κ is either infinite or even. If κ is finite and even, then this action is unique.*

I believe it should be possible to complete the classification as follows:

Conjecture 1.8. Let T be an $\mathbb{R}$-tree such that $|T| = 2^{\aleph_0}$. There exists a free transitive group action on T if and only if at least one of the following hold:

- The valence of T is $2^{\aleph_0}$.
- $T = T_\kappa^{[1]}$, where κ is either infinite or even.

1.3. **Products of $\mathbb{R}$-trees.** Products of locally finite simplicial trees provide an important source of examples in geometric group theory. Groups which act freely and transitively on the vertex sets of such spaces– sometimes called BMW groups – were used by Wise to provide the first example of a CAT(0) group which is not residually finite [Wis96], and by Burger–Mozes to provide the first examples of simple groups of the form $A *_C B$, where A, B, C are free groups of finite rank and C has finite index in both A and B [BM00]. Each of these examples involves a group G acting freely and vertex-transitively on a product of locally finite simplicial trees X such that, for any subgroup $H \leqslant G$ which splits non-trivially as a direct product, the induced action of H on X is not cocompact. Such BMW groups are called irreducible.

In [Azu25], I proved that this phenomenon persists in the continuous setting:

Corollary 1.9 ([Azu25, Corollary D]). *Let X be the ℓ^1 product of two universal $\mathbb{R}$-trees with valence $2^{\aleph_0}$. There exists a group G acting freely and transitively on X such that, for any subgroup $H \leqslant G$ which splits non-trivially as a direct product, the induced action of H on X is not cobounded (in particular, not cocompact).*

While the group G in the above theorem is irreducible in a strong sense, it does contain some very large normal subgroups and in particular it is not 'virtually simple' in any reasonable sense. This raises the following question:

Question 1.10 ([Azu25, Question 1.1]). Does there exist a simple group G acting freely and coboundedly on the ℓ^1 product of two $\mathbb{R}$-trees X?

A more general construction of groups acting on products of trees leads to the following analogue of Theorem 1.3:

Let $N \in \{\mathbb{N}\} \cup \{\{1, \dots, n\} : n \in \mathbb{N}\}$. Let $\mathrm{Sub}_D(\mathbf{R})$ denote the set of dense subgroups of $\mathbf{R}$ and let $\overline{\mathrm{Sub}}_D(\mathbf{R})$ be the quotient of $\mathrm{Sub}_D(\mathbf{R})$ under linear isometries of $\mathbf{R}$.

Theorem 1.11 ([Azu25, Theorem F]). *Let $\iota : \overline{\mathrm{Sub}_D}(\mathbf{R}) \to \mathcal{K}$ be any map which is supported on $\leqslant 2^{\aleph_0}$ elements of $\overline{\mathrm{Sub}_D}(\mathbf{R})$. Then there exists a group G, which acts freely and transitively on the ℓ^1 product S of $|N|$ universal $\mathbb{R}$-trees with valence $2^{\aleph_0}$, such that: for each $H \in \mathrm{Sub}_D(H)$, the cardinality of the set of orbits of maximal flats $F \subseteq S$ such that $\mathrm{Stab}_G(F) \curvearrowright F$ is isomorphic to $H \curvearrowright \mathbf{R}$ is $\iota([H])$.*

One of the principal problems in geometric group theory is to what extent good geometric properties of groups imply a property called *residual finiteness*, which means that the trivial subgroup can be written as an intersection of finite index subgroups. This is a rather subtle problem, and this subtlety is witnessed very well by groups with the geometry of a product of two simplicial trees, which can sometimes fail to be residually finite via the work of Wise and Burger–Mozes. An important question, first posed by Gromov in [Gro87], is that of whether all hyperbolic groups are residually finite. This question has been studied by numerous people since then and inspired a great deal of research, but it remains open. A possible road to finding a hyperbolic group which is not residually finite could lie in producing hyperbolic subgroups of groups which act freely and transitively on a product real trees, with the added flexibility of the real trees providing the possibility of creating hyperbolicity in a way that is impossible in products of simplicial trees, and the work of Wise and Burger–Mozes providing inspiration for ways to prevent residual finiteness in such groups.

1.4. **Finite rank median spaces.** Most of the results I have mentioned so far rely on a new method for constructing groups which are equipped with left-invariant median structures, and often left-invariant median metrics ([Azu25, Section 3]). Even among finitely generated groups, new sources of actions on median spaces are not very common so these are natural candidates to answer remaining open questions about the nature of finitely generated groups which act on median spaces. One such question that I am interested in needs a little introduction:

It was proven in [CDH10] that a locally compact second countable group G has Kazhdan's Property (T) if and only if it does not admit an action on a median space without a global fixed point, and G has the Haagerup property if and only if it admits a proper discontinuous action on a median space. A significant amount of work has been devoted to understanding intermediate fixed properties, i.e. which groups can act without fixed points on *some* median spaces but not those in a specific subclass. Subclasses which have been used for this purpose include finite rank median spaces (*rank* is a notion of dimension which depends only on the median structure) and CAT(0) cube complexes (which can be thought of as discrete median spaces). However the precise relationship between the classes of groups admitting actions on these spaces remains mysterious. While it is known that there are groups which act geometrically on infinite rank median spaces but not CAT(0) cube complexes (of any dimension) or finite rank median spaces [CD23; CFI16; Fio19], the following remains open:

Question 1.12. Does there exist a group G which admits a geometric action on a finite rank median space but does not act geometrically on any (finite dimensional) CAT(0) cube complex?

2. Subgroup structure

The second branch of my research deals with subgroups of finitely generated groups. One part of my work in this area is about the *space of subgroups* of finitely generated groups. Here one considers the entire set of subgroups of a finitely generated group as a topological space. My work on this space with Damien Gaboriau is the subject of Subsections 2.1 and 2.2. Working on the space of subgroups has led me to some interesting questions regarding the relationship between the structure of a subgroup embedding $H \hookrightarrow G$ and the geometry of the quotient space $H\backslash G$. A result in this direction is discussed in Subsection 2.3, and some further questions and conjectures pertaining to this are discussed in Subsection 2.4.

2.1. **The space of subgroups.** One approach to understanding the subgroups of a group is to study its space of subgroups. The set of subgroups of a finitely generated group G is equipped with a natural topology, known as the Chabauty topology, making it a compact, metrisable, totally disconnected space denoted $\mathrm{Sub}(G)$. The natural action of G by conjugation on $\mathrm{Sub}(G)$ is continuous. The Cantor–Bendixson Theorem implies that such a space admits a unique decomposition $\mathrm{Sub}(G) = \mathcal{K}(G) \sqcup C$, where $\mathcal{K}(G)$ is perfect (i.e. closed without isolated points) and C is countable. The space $\mathcal{K}(G)$ is called the *perfect kernel* of G and, when it is non-empty, it provides a natural action of G on a standard Borel space which witnesses much of the topological and ergodic behaviour of G. As such, actions on these spaces have received a lot of attention (see e.g. [AGV14; GW14; SZ94]) and it has been explicitly described for several classes of groups, including free groups, countable abelian groups [CGP10], lamplighter groups [GK14], Baumslag–Solitar groups [Car+25], generalised Baumslag-Solitar groups [Bon24] and some classes of branch groups [SW20]. In joint work with Damien Gaboriau, we significantly expanded the class of groups for which this space is known:

Corollary 2.1 ([AG25])**.** *Let G be a finitely generated group such that G is either:*

- *non-elementary hyperbolic and locally quasi-convex;*
- *a non-abelian limit group;*
- *a non-elementary free product;*
- *the fundamental group of a graph of free groups with cyclic edge groups such that at least one vertex group is non-cyclic.*

Then the perfect kernel $\mathcal{K}(G)$ of G is the set of infinite index subgroups of G.

If in addition G does not contain any non-trivial finite normal subgroups, then the action of G on $\mathcal{K}(G)$ by conjugation is highly topologically transitive.

The above list is a consequence of several independent theorems but there is a common theme to the proofs. The idea is that, if a group G acts 'sufficiently nicely' on a hyperbolic space (the exact conditions on the action depend on the space), and a subgroup $H \leqslant G$ stabilises a proper subspace which is convex, or *quasi-convex*, then it is often possible to find an element $g \in G$ which is "transverse to H". This usually amounts to saying that $\langle H, g\rangle \leqslant G$ splits as an amalgamated free product $H *_C \langle g\rangle$, where C is finite. Results of this kind are easiest to prove when the hyperbolic space is a tree, but, if the action is good enough, also hold for hyperbolic spaces [Gro87; Arz01; Git99; Mar09]. We use this structure to show that certain classes of subgroups in $\mathrm{Sub}(G)$ are perfect in the Chabauty topology. The proofs of topological transitivity and high topological transitivity use more involved versions of this idea, but in each case the key ingredient is convexity.

The behaviour of stabilisers of convex subspaces is easiest to control in the case of trees, so in this case the conditions we need to impose are very weak:

Theorem 2.2 ([AG25, Theorem H])**.** *Assume $G \curvearrowright T$ is a minimal and irreducible action on a tree. Assume that there are two edges f_1, f_2 of T such that $\mathrm{Stab}_G(f_1) \cap \mathrm{Stab}_G(f_2)$ is finite. Then*

$$B := \overline{\{H \leqslant G : H\backslash T \text{ is infinite }\}} \subseteq \mathcal{K}(G).$$

If, in addition, the kernel of the action $G \curvearrowright T$ is trivial then the action of G on B is highly topologically transitive.

In the more general context of hyperbolic spaces, we need to assume stronger conditions on the action:

Theorem 2.3 ([AG25, Theorem A])**.** *Let G be a non-elementary hyperbolic group. Then*

$$B = \overline{\{H \leqslant G : [G : H] = \infty \text{ and } H \text{ is quasi-convex}\}} \subseteq \mathcal{K}(G).$$

If in addition G has no non-trivial finite normal subgroups then the action of G on B is highly topologically transitive.

In many cases, including those mentioned in Corollary 2.1, the above two results show that the perfect kernel is the set of infinite index subgroups of G. However, as demonstrated below, this is not always the case, even for some very well-behaved hyperbolic groups:

Theorem 2.4 ([AG25, Theorem C])**.** *If G is the fundamental group of a closed hyperbolic 3-manifold then $\mathcal{K}(G)$ is the set of quasiconvex subgroups of G and the action of G on $\mathcal{K}(G)$ by conjugation is highly topologically transitive.*

The above theorem makes use of a remarkable fact about hyperbolic 3-manifold groups which is a consequence of Canary's Covering Theorem [Can96] and the Tameness Theorem [Ago04; CG06]: *Every finitely generated subgroup of a hyperbolic 3-manifold group G is either quasi-convex in G or a virtual fiber subgroup.* If $G = \pi_1(M)$ for some 3-manifold M, a virtual fiber subgroup of G is a subgroup $H \leqslant G$ which is the fundamental group of a surface Σ such that M admits a finite cover which fibers over the circle with fiber Σ. This is the motivating example of a more general notion which is defined as follows.

Definition 2.5. Let G be a finitely generated group. A subgroup $H \leqslant G$ is a *virtual fiber subgroup* if it is finitely generated and there are finite index subgroups $H' \leqslant H$ and $G' \leqslant G$ and a short exact sequence

$$1 \longrightarrow H' \longrightarrow G' \longrightarrow \mathbb{Z} \longrightarrow 1.$$

It is not hard to see that virtual fiber subgroups in the geometric sense are virtual fiber subgroups in the algebraic sense. It is a result due to Stallings that the converse also holds for subgroups of 3-manifold groups [Sta61].

All virtual fiber subgroups $H \leqslant G$ (in either sense) satisfy a geometric property of a different nature. Any finite generating set of G defines a word metric on it which is invariant by left multiplication. This metric depends on the choice of finite generating set but only up to *quasi-isometry* (i.e. the coarse geometry does not change). Therefore, in what follows, we fix a word metric on G. The quotient $H\backslash G$ is then equipped with a natural metric. If H is a virtual fiber subgroup then $H\backslash G$ is quasi-isometric to $\mathbb{Z}$ or $\mathbb{N}$, equipped with the usual metric. A partial converse to this is discussed in Section 2.3. This is the property we use to prove the second half of Theorem 2.4:

Proposition 2.6 ([AG25, Proposition 3.15])**.** *Let $H \leqslant G$ be a pair of finitely generated groups. If the quotient space $H\backslash G$ is quasi-isometric to a $\mathbb{Z}$ or $\mathbb{N}$ then there are at most countably many intermediate subgroups $H \leqslant K \leqslant G$. In particular, $H \notin \mathcal{K}(G)$.*

In [Azu24], I introduced the notion of narrow graphs (which includes all uniformly locally finite, unbounded graphs with linear growth and all quasi-trees with finitely many ends) and proved that the above theorem holds for all $H \leqslant G$ where $H\backslash G$ is narrow. I will discuss this property and the result which motivates it in Section 2.3.

2.2. **Amenable actions.** One consequence of being able to explicitly describe a large invariant subspace of the space of subgroups which is equipped with a topologically transitive action, is that it can be used to deduce the existence of certain actions.

The notion of amenable actions was introduced by von Neumann in response to the Banach-Tarski paradox. These are precisely the actions which do not admit paradoxical decompositions. Since then, amenable groups (i.e. groups which admit free amenable actions) have been very well studied. The class of groups which admit 'interesting' (but not free) amenable actions is still however much less well understood. In [GM07], Glasner–Monod introduced the class $\mathcal{A}$ of countably infinite groups which admit transitive faithful and amenable actions. This class was already known to contain all finite rank free groups by a result of van Douwen [Dou90], and they showed that it contains 'most' free products. Since then Moon showed that this class also includes surface groups and hyperbolic 3-manifold groups [Moo10] and Moon in [Moo11a; Moo11b] and Fima in [Fim14] showed that it includes some instances of amalgamated free products and HNN-extension over finite or amenable groups. Using our results about the space of subgroups (and some CAT(0) cube complex arguments) Damien Gaboriau and I were able to significantly expand this list:

Theorem 2.7 ([AG25, Section 8])**.** *Let G be a finitely generated infinite group such that at least one of the following hold:*

- *G is virtually compact special in the sense of Haglund–Wise [HW08].*
  This class includes in particular:
  - *Right-angled Artin groups;*
  - *finitely presented C'(1/6) small cancellation groups;*
  - *Right-angled and hyperbolic Coxeter groups;*
  - *One-relator groups with torsion;*
  - *Limit groups;*
  - *Random groups at density $d < 1/6$;*
  - *Hyperbolic-by-cyclic groups which are themselves hyperbolic.*
- *G is an infinitely ended group without a non-trivial finite normal subgroup and with some amenable action with infinite orbits. For instance, any HNN-extension G of any infinite group H over a pair of finite subgroups with trivial intersection.*
- *G is the fundamental group of any non-tree graph of groups where some edge group is malnormal in an incident vertex group*
- *G is the fundamental group of a finite graphs of free groups with cyclic edge groups such that at least one vertex group is non-cyclic.*

Then $G \in \mathcal{A}$.

2.3. **Virtual fiber subgroups via narrow Schreier graphs.** As previously mentioned, knowing that, for a pair of finitely generated groups $H \leqslant G$, the quotient space $H\backslash G$ is quasi-isometric to a line or quasi-line can be very revealing. Geometrically speaking, it is an extremely strong condition, and therefore it is reasonable to ask if there are any examples satisfying this property which are not virtual fiber subgroups (in the sense of Definition 2.5). It turns out there are such examples; Houghton produced a family of groups in [Hou74] which can be used to produce such pairs (see [Azu24, Section 6]). However, the theorem below shows that the class of pairs of groups which can satisfy this geometric property without being virtual fiber subgroups is extremely limited.

The *number of ends* of a space is a quasi-isometry invariant which can be interpreted as the number of distinct ways of going to infinity in the space.

Theorem 2.8 ([Azu24, Theorem 1.1])**.** *Let $H \leqslant G$ be a pair of finitely generated groups such that the quotient space $H\backslash G$ has at least two ends and is narrow (e.g. it has linear growth or is quasi-isometric to a finitely ended tree). Then the following are equivalent:*

i. *H is a virtual fiber subgroup;*
ii. *H is separable in G (i.e. $H = \cap_n K_n$, where each K_n is a finite index subgroup of G);*
iii. *H has bounded packing in G;*
iv. *there are infinitely many double cosets of H in G.*

Narrowness is a property, defined in the same paper, which bounds the number of pairwise distinct, coarsely connected infinite subspaces one can find in a graph, at every scale. It is mostly applied to the images of cosets in $H\backslash G$ and allows one to use the pigeonhole principle to bound certain hyperbolic-like behaviours in certain spaces that G acts on.

More precisely, the proof of this theorem relies on CAT(0) cube complexes. Those unfamiliar with these should think of them as higher dimensional analogues of simplicial trees (built out of cubes) where the property that each midpoint of an edge separates a tree into two connected components is replaced with the property that *each edge determines a locally codimension-1 subspace called a hyperplane, which separates the cube complex into two connected components*. The condition that $H\backslash G$ has at least two ends implies, by a theorem of Sageev [Sag95], that G acts on a CAT(0) cube complex C with hyperplane stabiliser commensurable with H. The relationship between the geometry of the the quotient space $H\backslash G$ and the cube complex C is typically not particularly explicit. In fact, there are a number of choices involved in Sageev's construction, which can drastically change the structure of the resulting cube complex. However the bounded packing property, which was introduced by Hruska and Wise in [HW09], always ensures that the resulting cube complex is finite dimensional. This provides access to tools developed by Caprace and Sageev in [CS11]. Using these tools, the narrowness assumption can be used to show that the cube complex C has no *facing triple of hyperplanes* (by [Hag22, Corollary 3.9], this is equivalent to saying that C isometrically embeds in $\mathbb{R}^n$, equipped with the ℓ^1 metric, for some n.) From there, the theorem follows from the following result from the same paper:

Proposition 2.9 ([Azu24, Proposition 1.12])**.** *Let Γ be a group and suppose that Γ acts essentially on a finite dimensional CAT(0) cube complex Y which has no facing triples. Then for any hyperplane $\mathfrak{h}$ in Y, there are finite index subgroups $K \leqslant \mathrm{Stab}_\Gamma(\mathfrak{h})$ and $\Lambda \leqslant \Gamma$ such that $K \trianglelefteq \Lambda$ and $\Lambda/K = \mathbb{Z}$.*

Let me end by mentioning that this theorem is far from isolated in the literature. When Hopf defined the notion of ends of a finitely generated group in [Hop44], he proved that a finitely generated group is two-ended if and only if it is virtually cyclic; in other words, a finitely generated normal subgroup $N \trianglelefteq G$ is a virtual fiber subgroup if and only if the quotient $N\backslash G$ is 2-ended. When Houghton introduced the number of relative ends of a pair of groups $H \leqslant G$, he weakened the normality assumption: if $H \leqslant G$ is finitely generated, $H\backslash G$ is two-ended and H has infinite index in its normaliser then H is a virtual fiber subgroup. Finally, a similar but logically independent result due to Kropholler–Roller [KR89] states that, if a finitely generated pair $H \leqslant G$ has two filtered ends and H has infinite index in its commensurator, then H is a virtual fiber subgroup.

2.4. **Right-angled Artin groups.** Right-angled Artin groups, or RAAGs, are an extremely important class of groups with a deceivingly simple definition: they are finitely generated groups, where some of the generators commute. Another way of phrasing this which turns out to be useful is the following. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite simple graph. The RAAG with defining graph Γ is the group:

$$A_\Gamma := \langle V(\Gamma) \mid [a,b] \ \forall \ \{a,b\} \in E(\Gamma)\rangle.$$

The reason this seemingly innocuous class of groups is so important boils down to two facts: first, RAAGs possess a number of powerful geometric and algebraic properties, and second, they admit a surprisingly large class of subgroups. Examples of the properties which these groups enjoy include being linear over $\mathbb{Z}$, and in particular residually finite, having separable convex cocompact subgroups, and being RFRS (a technical residual condition introduced by Ian Agol in [Ago08] which makes it easier to determine whether a group virtually algebraically fibers). Each of these properties are highly non-trivial and remain unknown for many classes of groups and, crucially, they are closed under passing to subgroups.

In [HW08], Haglund and Wise gave a geometric characterisation of being the subgroup of a RAAG which consists of being the fundamental group of a non-positively curved cube complex which does not contain any 'pathological hyperplane configurations'. Following on from this, Wise led a program to prove that a great deal of hyperbolic groups virtually embed into RAAG's, with the ultimate aim of proving the vitual Haken conjecture by proving that any hyperbolic 3-manifold group virtually embeds in a RAAG. This program was realised through numerous ground-breaking contributions [BW12; KM12; Wis21], the final step of which is the following astonishing result of Agol [Ago13]: *every hyperbolic group which acts properly and cocompactly on a CAT(0) cube complex is virtually a (convex cocompact) subgroup of a RAAG.* Using his earlier work on RFRS groups, this result not only proved the virtual Haken conjecture, but also the virtual fibering conjecture.

In light of the pivotal role that RAAG's, and particularly their subgroups, play, understanding their space of subgroups is of particular interest. For reasons I will outline below, I believe the following should be true:

Conjecture 2.10. Let A_Γ be an irreducible RAAG. Then

$$\mathrm{Sub}_{NE}(A_\Gamma) := \{H \leqslant A_\Gamma : H \text{ stabilises a proper convex subspace of } A_\Gamma\} = \mathcal{K}(A_\Gamma).$$

The inclusion of $\mathrm{Sub}_{NE}(A_\Gamma)$ into $\mathcal{K}(A_\Gamma)$ is essentially a corollary of Theorem 2.2, using some CAT(0) cube complex geometry (see [AG25, Lemma 8.5]). The other direction presents new challenges as it requires one to understand the subgroups of A_Γ which act on it, not convex cocompactly (which is where the most established tools are available), or literally cocompactly, but only *essentially*. This is a term relating to the fact that A_Γ is the 1-skeleton of a CAT(0) cube complex S_Γ. The subgroups one needs to understand to prove the above conjecture are those whose orbit intersects every halfspace of this CAT(0) cube complex, but where the quotient is still infinite. While much weaker than a cocompact action, an essential action on a CAT(0) cube complex is still a powerful condition, and I am hopeful that, with careful analysis, it should impose sufficiently strong restrictions on the quotient to prove that these subgroups do not belong to the perfect kernel, using methods inspired by Proposition 2.6.

My strategy for tackling this conjecture also ties in with the following question, which is reminiscent of Theorem 2.8 and is closely related to work of Rubenstein–Wang on horizontal subgroups of graph manifolds [RW98]:

Question 2.11. Let A_Γ be a 2-dimensional RAAG and let $H \leqslant A_\Gamma$ be a finitely generated, infinite index subgroup which acts essentially on A_Γ. Are the following equivalent?

- H is a virtual fiber subgroup;
- H is separable in A_Γ;
- $H\backslash A_\Gamma$ is quasi-isometric to $\mathbb{Z}$ or $\mathbb{N}$;
- there are infinitely many double cosets of H in A_Γ.

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